MT305.01: Advanced Calculus for Science Majors
Examination 1
Answers

1. (*18 points*) Suppose that $2\dfrac{dx}{dt} + x + 1 = 0$, and $x(0) = 3$. Find an explicit expression for x in terms of t.

Answer: We have

$$\begin{aligned} 2\frac{dx}{dt} &= -x - 1 \\ \int \frac{dx}{x+1} &= -\frac{1}{2}\int dt \\ \log(x+1) &= -\frac{t}{2} + C \\ x + 1 &= De^{-t/2} \\ x &= De^{-t/2} - 1 \end{aligned}$$

Now, substitution of $t = 0$ and $x = 3$ yields $D = 4$, so the complete solution is $x = 4e^{-t/2} - 1$.

2. (*18 points*) Suppose that $\dfrac{dy}{dx} = \sin(y - x)$, and $y(0) = 1$. Find an implicit definition for y in terms of x.

Answer: Write $u = y - x$, so that $y = x + u$ and $\dfrac{dy}{dx} = 1 + \dfrac{du}{dx}$. The differential equation now becomes $1 + \dfrac{du}{dx} = \sin u$, or

$$\begin{aligned} \frac{du}{dx} &= \sin u - 1 \\ \int dx &= -\int \frac{du}{1 - \sin u} = -\int \left(\frac{1 + \sin u}{1 + \sin u}\right) \frac{dx}{1 - \sin u} = -\int \frac{(1 + \sin u)dx}{1 - \sin^2 u} \\ &= -\int \frac{(1 + \sin u)dx}{\cos^2 u} = -\int \left(\frac{1}{\cos^2 u} + \frac{\sin u}{\cos^2 u}\right) du \\ &= -\tan u - \sec u \end{aligned}$$

Therefore, the equation is $x + C = -\tan u - \sec u = -\tan(y - x) - \sec(y - x)$. Substitute $x = 0$ and $y = 1$, and we get $C = -\tan 1 - \sec 1$, and therefore the equation is $x - \tan 1 - \sec 1 = \tan(y - x) - \sec(y - x)$.

3. (*18 points*) Suppose that $\dfrac{dy}{dx} = \dfrac{x^2 + y^2}{2xy}$ and $y(1) = 2$. Find an explicit expression for y in terms of x.

Answer: The equation is the quotient of homogeneous polynomials, so we substitute $y = ux$ and $\dfrac{dy}{dx} = u + x\dfrac{du}{dx}$. We get

$$\begin{aligned} u + x\frac{du}{dx} &= \frac{x^2 + x^2u^2}{2x^2u} = \frac{1 + u^2}{2u} \\ x\frac{du}{dx} &= \frac{1 + u^2}{2u} - u = \frac{1 - u^2}{2u} \\ \int \frac{2u}{1 - u^2} du &= \int \frac{dx}{x} \\ \int \frac{2u}{u^2 - 1} du &= -\int \frac{dx}{x} \\ \log(u^2 - 1) &= -\log x + C \\ u^2 - 1 &= \frac{D}{x} \\ u^2 &= 1 + \frac{D}{x} = \frac{D + x}{x} \end{aligned}$$

$$\frac{y^2}{x^2} = \frac{D+x}{x}$$
$$y^2 = x^2 + Dx$$
$$y = \sqrt{x^2 + Dx}$$

Now, we substitute $x = 1$ and $y = 2$ to get $\sqrt{1+D} = 2$, or $1 + D = 4$, and so $D = 3$. The solution is $y = \sqrt{x^2 + 3x}$.

4. (*20 points*) Suppose that $\frac{dy}{dt} + y = ty^2$, with $y(0) = 1$. Make the substitution $x = y^{-1}$ and solve the differential equation, finding an explicit expression for y in terms of t.

Answer: If $x = y^{-1}$, then $y = x^{-1}$, and so $\frac{dy}{dt} = \frac{d(x^{-1})}{dt} = -x^{-2}\frac{dx}{dt}$. The equation becomes

$$-x^{-2}\frac{dx}{dt} + x^{-1} = tx^{-2}$$
$$\frac{dx}{dt} - x = -t$$

We use e^{-t} as an integrating factor:

$$e^{-t}\frac{dx}{dt} - xe^{-t} = -te^{-t}$$
$$\frac{d}{dt}\left(xe^{-t}\right) = -te^{-t}$$
$$xe^{-t} = te^{-t} + e^{-t} + C$$
$$x = t + 1 + Ce^t$$
$$y = \frac{1}{t+1+Ce^t}$$

Substitute $t = 0$ and $y = 1$, and we get $1 = \frac{1}{1+C}$, so $C = 0$, and we have $y = \frac{1}{t+1}$.

5. (*26 points*) Solve the differential equation $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = e^{-2x}$ for y as a function of x.

Answer: We start by solving the auxiliary equation $m^2 + 4m + 13 = 0$. We learn that $m = \frac{-4 \pm \sqrt{16-52}}{2} = \frac{-4 \pm \sqrt{-36}}{2} = \frac{-4 \pm 6i}{2} = -2 \pm 3i$. Therefore, $y_h = e^{-2x}(A\cos 3x + B\sin 3x)$.

Use undetermined coefficients to compute $y_p = Ce^{-2x}$. We compute $y_p' = -2Ce^{-2x}$ and $y_p'' = 4Ce^{-2x}$. We have $4Ce^{-2x} + 4(-2)Ce^{-2x} + 13Ce^{-2x} = e^{-2x}$. Multiply by e^{2x}, and the result is $9C = 1$, or $C = \frac{1}{9}$. Therefore, the general solution is $y = e^{-2x}(\frac{1}{9} + A\cos 3x + B\sin 3x)$.

Grade	Number of people
90	1
87	1
82	1
77	2
76	1
72	1
71	1
69	1
64	1
51	2
39	1

Mean: 69.69
Standard deviation: 14.39