MT305.01: Advanced Calculus for Science Majors
Examination 2
Answers

1. (*20 points*) Solve

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = 2\sin 2t.$$

Answer: This problem can be done many ways. Here's how to do it using undetermined coefficients.

We solve $x'' + 3x' + 2x = 0$ by solving the auxiliary equation $r^2 + 3r + 2 = 0$ and finding that $r = -1$ and $r = -2$ are the two solutions. Therefore, the solution to the homogeneous equation is $x = Ae^{-t} + Be^{-2t}$.

We guess at a solution of the form $x = C\sin 2t + D\cos 2t$, and then $x' = 2C\cos 2t - 2D\sin 2t$ and $x'' = -4C\sin 2t - 4D\cos 2t$. We have

$$x'' + 3x' + 2x = (-4C\sin 2t - 4D\cos 2t) + 3(2C\cos 2t - 2D\sin 2t) + 2(C\sin 2t + D\cos 2t) = 2\sin 2t.$$

This leads to the two equations

$$\begin{aligned} -2C - 6D &= 2 \\ 6C - 2D &= 0 \end{aligned}$$

Multiply the first equation by 3 and add, and we get $-20D = 6$, so $D = -\frac{3}{10}$ and $C = -\frac{1}{10}$.

Therefore, $x = Ae^{-t} + Be^{-2t} - \frac{1}{10}\sin 2t - \frac{3}{10}\cos 2t$.

2. (*20 points*) Solve

$$\frac{d^2y}{dt^2} + 9y = \begin{cases} 0 & 0 \le t < 1 \\ 9(t-1) & 1 \le t < 2 \\ 9 & 2 \le t \end{cases} \qquad y(0) = 0, \quad y'(0) = 0.$$

Answer: The simplest way to do this problem is using Laplace transforms. We have to express the right-hand side of the differential equations using step functions, and the result is

$$(\mathscr{U}(t-1) - \mathscr{U}(t-2))(9)(t-1) + \mathscr{U}(t-2)9 = \mathscr{U}(t-1)9(t-1) - \mathscr{U}(t-2)(9)(t-2)$$

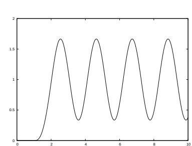
So we have

$$\begin{aligned} y'' + 9y &= \mathscr{U}(t-1)9(t-1) - \mathscr{U}(t-2)(9)(t-2) \\ \mathscr{L}(y'') + 9\mathscr{L}(y) &= 9\mathscr{L}(\mathscr{U}(t-1)(t-1)) - 9\mathscr{L}(\mathscr{U}(t-2)(t-2)) \\ s^2Y + 9Y &= 9\left(\frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s^2}\right) \\ Y &= 9\left(\frac{e^{-s}}{s^2(s^2+9)} - \frac{e^{-2s}}{s^2(s^2+9)}\right) \end{aligned}$$

Substitute $\dfrac{1}{s^2(s^2+9)} = \dfrac{1}{9}\left(\dfrac{1}{s^2} - \dfrac{1}{s^2+9}\right)$:

$$\begin{aligned} Y &= e^{-s}\left(\frac{1}{s^2} - \frac{1}{s^2+9}\right) - e^{-2s}\left(\frac{1}{s^2} - \frac{1}{s^2+9}\right) \\ y &= \mathscr{U}(t-1)\left((t-1) - \frac{1}{3}\sin(3(t-1))\right) - \mathscr{U}(t-2)\left((t-2) - \frac{1}{3}\sin(3(t-2))\right). \end{aligned}$$

The graph is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3. (*20 points*) Solve

$$\frac{d^2y}{dt^2} + y = \sec t.$$

Answer: The homogeneous equation has general solution $y = A\cos t + B\sin t$. Set $y_1 = \cos t$, $y_2 = \sin t$, $y_1' = -\sin t$, and $y_2' = \cos t$, and variation of parameters tells us that $y_p = u_1y_1 + u_2y_2$, with

$$y_1u_1' + y_2u_2' = 0$$
$$y_1'u_1' + y_2'u_2' = \sec t$$

We have

$$(-\sin t)u_1' + (\cos t)u_2' = \sec t$$
$$(\cos t)u_1' + (\sin t)u_2' = 0$$

Multiply the first equation by $\cos t$ and the second by $\sin t$:

$$(-\sin t\cos t)u_1' + (\cos^2 t)u_2' = 1$$
$$(\sin t\cos t)u_1' + (\sin^2 t)u_2' = 0$$

We get $u_2' = 1$, so $u_2 = t$. If $u_2' = 1$, then $u_1' = -(\sin t)u_2'/\cos t = -\sin t/\cos t = -\tan t$, and then $u_1 = \log(\cos t)$.

The general solution is $y = \cos t(\log(\cos t)) + t\sin t + A\cos t + B\sin t$.

4. (*20 points*) Use Laplace transforms to show that

$$(\cos t) * \left(\frac{1}{k}\sin(kt)\right) = (\sin t) * (\cos kt).$$

You may assume that if $\mathscr{L}(f) = \mathscr{L}(g)$ and f and g are continuous, then $f = g$.
Answer: We know that

$$\mathscr{L}\left((\cos t) * \left(\frac{1}{k}\sin(kt)\right)\right) = \mathscr{L}(\cos t)\mathscr{L}\left(\frac{1}{k}\sin(kt)\right) = \frac{s}{s^2+1}\left(\frac{1}{s^2+k^2}\right) = \frac{s}{(s^2+1)(s^2+k^2)}$$
$$\mathscr{L}\big((\sin t) * \cos(kt)\big) = \mathscr{L}(\sin t)\mathscr{L}(\cos(kt)) = \left(\frac{1}{s^2+1}\right)\left(\frac{s}{s^2+k^2}\right) = \frac{s}{(s^2+1)(s^2+k^2)}$$

The two functions have the same Laplace transforms and are continuous, and hence must be the same.

5. (*20 points*) Solve

$$t^2\frac{d^2x}{dt^2} - 5t\frac{dx}{dt} + 11x = 0.$$

Answer: We start by solving $t^2x'' - 5tx' + 11x = 0$ by letting $x = t^m$, $x' = mt^{m-1}$, and $x'' = m(m-1)t^{m-2}$. We have $m(m-1) - 5m + 11 = 0$, or $m^2 - 6m + 11 = 0$, and then $m = \frac{6 \pm \sqrt{36-44}}{2} = \frac{6 \pm \sqrt{-8}}{2} = 3 \pm \sqrt{-2}$. Our recipe now tells us that $x = t^3\left(A\cos\sqrt{2}(\log x) + B\sin\sqrt{2}(\log x)\right)$.

Grade	Number of people
94	1
90	1
83	1
79	1
77	1
75	2
72	1
70	1
69	1
55	1
50	2

Mean: 72.23
Standard deviation: 13.27