1. (25 points) Solve the differential equation

\[
\frac{d^2 y}{dt^2} + 4y = \begin{cases} 
0 & 0 \leq t < \pi \\
1 & \pi \leq t < 2\pi \\
0 & 2\pi \leq t 
\end{cases}
\]

with initial conditions \(y(0) = 1\) and \(y'(0) = 1\).

**Answer:** Rewrite the equation in the form \(y'' + 4y = \mathcal{U}(t - \pi) - \mathcal{U}(t - 2\pi)\). Taking the Laplace transform now yields:

\[
s^2Y - s - 1 + 4Y = \frac{e^{-\pi s} - e^{-2\pi s}}{s}
\]

which can be rearranged to give

\[
Y = \frac{s + 1}{s^2 + 4} + \frac{e^{-\pi s} - e^{-2\pi s}}{s(s^2 + 4)}
\]

Partial fractions yields

\[
\frac{1}{s(s^2 + 4)} = \frac{1}{4} \left( \frac{1}{s} - \frac{s}{s^2 + 4} \right)
\]

and so

\[
Y = \frac{s + 1}{s^2 + 4} + \left( \frac{e^{-\pi s} - e^{-2\pi s}}{4} \right) \left( \frac{1}{s} - \frac{s}{s^2 + 4} \right)
\]

Therefore,

\[
y = \mathcal{L}^{-1}(Y) = \mathcal{L}^{-1} \left( \frac{s + 1}{s^2 + 4} \right) + \mathcal{L}^{-1} \left( \frac{e^{-\pi s} - e^{-2\pi s}}{4} \right) \left( \frac{1}{s} - \frac{s}{s^2 + 4} \right)
\]

\[
= \mathcal{L}^{-1} \left( \frac{s}{s^2 + 4} \right) + \frac{1}{2} \mathcal{L}^{-1} \left( \frac{2}{s^2 + 4} \right) + \mathcal{L}^{-1} \left( \frac{e^{-\pi s}}{s^2 + 4} \right) - \mathcal{L}^{-1} \left( \frac{e^{-2\pi s}}{s^2 + 4} \right) + \mathcal{L}^{-1} \left( \frac{e^{-2\pi s}}{s^2 + 4} \right)
\]

\[
= \cos(2t) + \frac{1}{2} \sin(2t) + \frac{1}{4} \left( \mathcal{U}(t - \pi)(1 - \cos(2(t - \pi))) - \mathcal{U}(t - 2\pi)(1 - \cos(2(t - 2\pi))) \right)
\]

\[
= \cos(2t) + \frac{1}{2} \sin(2t) + \frac{1}{4} \left( \mathcal{U}(t - \pi)(1 - \cos(2t)) - \mathcal{U}(t - 2\pi)(1 - \cos(2t)) \right)
\]

\[
= \begin{cases} 
\cos(2t) + \frac{1}{2} \sin(2t) & 0 \leq t < \pi \\
\cos(2t) + \frac{1}{2} \sin(2t) + \frac{1}{4}(1 - \cos(2t)) & \pi \leq t < 2\pi \\
\cos(2t) + \frac{1}{2} \sin(2t) & 2\pi \leq t 
\end{cases}
\]

2. (25 points) Suppose we write the solution to the differential equation \(y'' + (x^2 + 1)y = 0\), with initial conditions \(y(0) = 1\) and \(y'(0) = 2\), in the form \(y = \sum_{n \geq 0} c_n x^n\). Write out the first 5 non-zero terms in the series expansion for \(y\).

**Answer:** We have \(y'' = \sum_{n \geq 0} (n-1)(n-2)c_n x^{n-2} = \sum_{n \geq 2}(n+2)(n+1)c_{n+2} x^n\) and \(x^2y = \sum_{n \geq 0} c_n x^{n+2} = \sum_{n \geq 2} c_{n-2} x^n\). So we have

\[
\sum_{n \geq 0} (n+2)(n+1)c_{n+2} x^n + \sum_{n \geq 2} c_{n-2} x^n + \sum_{n \geq 0} c_n x^n = 0.
\]

We are given \(c_0 = 1\) and \(c_1 = 2\). The constant term in the series expansion now gives \(2c_2 + c_0 = 0\), and so \(c_2 = -\frac{1}{2}\). The \(x\)-term gives \(6c_3 + c_1 = 0\), and so \(c_3 = -\frac{1}{3}\). Finally, the \(x^2\)-term gives \(12c_4 + c_0 + c_2 = 0\), and so \(c_4 = -\frac{1}{24}\). The first 5 non-zero terms are therefore \(1 + 2x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{24}\).
3. (25 points) The two solutions of the differential equation \(2xy'' + 5y' + xy = 0\) can both be written in the form \(y = x^r \sum_{n \geq 0} c_n x^n\), for two different values of \(r\), with \(c_0 = 1\). Compute the two possible values of \(r\).

**Answer:** We have

\[
y = \sum_{n \geq 0} c_n x^{n+r}
\]

\[
x y = \sum_{n \geq 0} c_n x^{n+r+1}
\]

\[
5y' = \sum_{n \geq 0} 5(n + r)c_n x^{n+r-1}
\]

\[
2xy'' = \sum_{n \geq 0} 2(n + r)(n + r - 1)c_n x^{n+r-1}
\]

\[
2xy'' + 5y' + xy = \sum_{n \geq 0} 2(n + r)(n + r - 1)c_n x^{n+r-1} + 5(n + r)c_n x^{n+r-1} + c_n x^{n+r+1} = 0
\]

The coefficient of \(x^{r-1}\) is \(c_0(2r(r - 1) + 5r)\), so we must have \(2r^2 + 3r = 0\). This gives \(r = 0\) and \(r = -\frac{3}{2}\) as the two possible values for \(r\).

4. (25 points) Write out the first 5 non-zero terms in the Fourier expansion of the function \(f(x) = |\sin x|\) for \(-\pi \leq x \leq \pi\).

**Answer:** The function \(f(x)\) is even, and so the only terms which appear in the Fourier expansion are the constant term and the ones involving cosines. We compute

\[
a_0 = \frac{2}{\pi} \int_0^\pi \sin x \, dx = \frac{4}{\pi}
\]

\[
a_1 = \frac{2}{\pi} \int_0^\pi \sin x \cos x \, dx = \frac{1}{\pi} \sin^2 x \bigg|_0^\pi = 0
\]

\[
a_n = \frac{2}{\pi} \int_0^\pi \sin x \cos nx \, dx = \left(\frac{2}{\pi}\right) \left(\frac{1}{n^2 - 1}\right) \left[n \sin x \sin nx + \cos x \cos nx\right]_0^\pi
\]

\[
= \left(\frac{2}{\pi}\right) \left(\frac{1}{n^2 - 1}\right) ((-1)^{n+1} - 1)
\]

and so \(a_2 = -\frac{4}{3\pi}\), \(a_3 = 0\), \(a_4 = -\frac{4}{15\pi}\), \(a_5 = 0\), \(a_6 = -\frac{4}{35\pi}\), \(a_7 = 0\), and \(a_8 = -\frac{4}{63\pi}\). We have

\[
|\sin x| = \frac{2}{\pi} - \frac{4}{\pi} \left(\frac{\cos 2x}{3} + \frac{\cos 4x}{15} + \frac{\cos 6x}{35} + \frac{\cos 8x}{63} + \cdots\right)
\]

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Mean: 67.46
Standard deviation: 11.58