MT305.01: Advanced Calculus for Science Majors
Examination 3
Answers

1. (*25 points*) Solve the differential equation

$$\frac{d^2y}{dt^2} + 4y = \begin{cases} 0 & 0 \le t < \pi \\ 1 & \pi \le t < 2\pi \\ 0 & 2\pi \le t \end{cases}$$

with initial conditions $y(0) = 1$ and $y'(0) = 1$.
Answer: Rewrite the equation in the form $y'' + 4y = \mathscr{U}(t-\pi) - \mathscr{U}(t-2\pi)$. Taking the Laplace transform now yields:

$$s^2Y - s - 1 + 4Y = \frac{e^{-\pi s} - e^{-2\pi s}}{s}$$

which can be rearranged to give

$$Y = \frac{s+1}{s^2+4} + \frac{e^{-\pi s} - e^{-2\pi s}}{s(s^2+4)}$$

Partial fractions yields

$$\frac{1}{s(s^2+4)} = \frac{1}{4}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)$$

and so

$$Y = \frac{s+1}{s^2+4} + \left(\frac{e^{-\pi s} - e^{-2\pi s}}{4}\right)\left(\frac{1}{s} - \frac{s}{s^2+4}\right)$$

Therefore,

$$\begin{aligned}
y = \mathscr{L}^{-1}(Y) &= \mathscr{L}^{-1}\left(\frac{s+1}{s^2+4}\right) + \mathscr{L}^{-1}\left(\left(\frac{e^{-\pi s} - e^{-2\pi s}}{4}\right)\left(\frac{1}{s} - \frac{s}{s^2+4}\right)\right) \\
&= \mathscr{L}^{-1}\left(\frac{s}{s^2+4}\right) + \frac{1}{2}\mathscr{L}^{-1}\left(\frac{2}{s^2+4}\right) + \mathscr{L}^{-1}\left(\left(\frac{e^{-\pi s} - e^{-2\pi s}}{4}\right)\left(\frac{1}{s} - \frac{s}{s^2+4}\right)\right) \\
&= \cos(2t) + \frac{1}{2}\sin(2t) + \frac{1}{4}\left(\mathscr{L}^{-1}\left(\frac{e^{-\pi s}}{s}\right) - \mathscr{L}^{-1}\left(\frac{e^{-\pi s}s}{s^2+4}\right) - \mathscr{L}^{-1}\left(\frac{e^{-2\pi s}}{s}\right) + \mathscr{L}^{-1}\left(\frac{e^{-2\pi s}s}{s^2+4}\right)\right) \\
&= \cos(2t) + \frac{1}{2}\sin(2t) + \frac{1}{4}\left(\mathscr{U}(t-\pi)(1-\cos(2(t-\pi))) - \mathscr{U}(t-2\pi)(1-\cos(2(t-2\pi)))\right) \\
&= \cos(2t) + \frac{1}{2}\sin(2t) + \frac{1}{4}\left(\mathscr{U}(t-\pi)(1-\cos(2t)) - \mathscr{U}(t-2\pi)(1-\cos(2t))\right) \\
&= \begin{cases} \cos(2t) + \frac{1}{2}\sin(2t) & 0 \le t < \pi \\ \cos(2t) + \frac{1}{2}\sin(2t) + \frac{1}{4}(1-\cos(2t)) & \pi \le t < 2\pi \\ \cos(2t) + \frac{1}{2}\sin(2t) & 2\pi \le t \end{cases}
\end{aligned}$$

2. (*25 points*) Suppose we write the solution to the differential equation $y'' + (x^2+1)y = 0$, with initial conditions $y(0) = 1$ and $y'(0) = 2$, in the form $y = \sum_{n\ge 0} c_n x^n$. Write out the first 5 non-zero terms in the series expansion for y.
Answer: We have $y'' = \sum_{n\ge 0} n(n-1)c_n x^{n-2} = \sum_{n\ge 0}(n+2)(n+1)c_{n+2}x^n$ and $x^2y = \sum_{n\ge 0} c_n x^{n+2} = \sum_{n\ge 2} c_{n-2}x^n$. So we have

$$\sum_{n\ge 0}(n+2)(n+1)c_{n+2}x^n + \sum_{n\ge 2} c_{n-2}x^n + \sum_{n\ge 0} c_n x^n = 0.$$

We are given $c_0 = 1$ and $c_1 = 2$. The constant term in the series expansion now gives $2c_2 + c_0 = 0$, and so $c_2 = -\frac{1}{2}$. The x-term gives $6c_3 + c_1 = 0$, and so $c_3 = -\frac{1}{3}$. Finally, the x^2-term gives $12c_4 + c_0 + c_2 = 0$, and so $c_4 = -\frac{1}{24}$. The first 5 non-zero terms are therefore $1 + 2x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{24}$.

3. (*25 points*) The two solutions of the differential equation $2xy'' + 5y' + xy = 0$ can both be written in the form $y = x^r \sum_{n\geq 0} c_n x^n$, for two different values of r, with $c_0 = 1$. Compute the two possible values of r.

Answer: We have

$$\begin{aligned} y &= \sum c_n x^{n+r} \\ xy &= \sum c_n x^{n+r+1} \\ 5y' &= \sum 5(n+r)c_n x^{n+r-1} \\ 2xy'' &= \sum 2(n+r)(n+r-1)c_n x^{n+r-1} \\ 2xy'' + 5y' + xy &= \sum 2(n+r)(n+r-1)c_n x^{n+r-1} + 5(n+r)c_n x^{n+r-1} + c_n x^{n+r+1} = 0 \end{aligned}$$

The coefficient of x^{r-1} is $c_0(2r(r-1)+5r)$, so we must have $2r^2 + 3r = 0$. This gives $r = 0$ and $r = -\frac{3}{2}$ as the two possible values for r.

4. (*25 points*) Write out the first 5 non-zero terms in the Fourier expansion of the function $f(x) = |\sin x|$ for $-\pi \leq x \leq \pi$.

Answer: The function $f(x)$ is even, and so the only terms which appear in the Fourier expansion are the constant term and the ones involving cosines. We compute

$$\begin{aligned} a_0 &= \frac{2}{\pi}\int_0^\pi \sin x \, dx = \frac{4}{\pi} \\ a_1 &= \frac{2}{\pi}\int_0^\pi \sin x \cos x \, dx = \frac{1}{\pi} \sin^2 x \Big|_0^\pi = 0 \\ a_n &= \frac{2}{\pi}\int_0^\pi \sin x \cos nx \, dx = \left(\frac{2}{\pi}\right)\left(\frac{1}{n^2-1}\right)\Big[n \sin x \sin nx + \cos x \cos nx\Big]_0^\pi \\ &= \left(\frac{2}{\pi}\right)\left(\frac{1}{n^2-1}\right)((-1)^{n+1} - 1) \end{aligned}$$

and so $a_2 = \dfrac{-4}{3\pi}$, $a_3 = 0$, $a_4 = \dfrac{-4}{15\pi}$, $a_5 = 0$, $a_6 = \dfrac{-4}{35\pi}$, $a_7 = 0$, and $a_8 = \dfrac{-4}{63\pi}$. We have

$$|\sin x| = \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{\cos 2x}{3} + \frac{\cos 4x}{15} + \frac{\cos 6x}{35} + \frac{\cos 8x}{63} + \cdots\right)$$

Grade	Number of people
85	1
78	1
77	1
75	3
68	1
65	1
64	2
60	1
48	1
43	1

Mean: 67.46
Standard deviation: 11.58