MT305.01: Advanced Calculus for Science Majors
Examination 4
May 2, 2012

Do all of your work in the blue booklets. Please label your answers clearly, as I will not have time to perform extensive searches for answers. No credit will be given for answers without explanations.

Cheating will result in a failing grade.

Calculators may not be used during this examination.

The problems are not arranged in order of increasing difficulty, so you might want to read all of them before beginning.

1. (*20 points*) The *Helmholtz equation* is

$$\nabla^2 u = -ku.$$

In spherical coordinates,

$$\nabla^2 u = \frac{\partial^2 u}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\left(\frac{\partial^2 u}{\partial \theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial u}{\partial \theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2}\right).$$

Write $u(\rho,\theta,\phi) = R(\rho)Y(\theta,\phi)$, and perform a separation of variables to get a partial differential equation for $Y(\theta,\phi)$ and an ordinary differential equation for $R(\rho)$. *You should not try to solve either equation.*

2. (*20 points*) Let n be a positive integer. Give the general solution of

$$t^2\frac{d^2y}{dt^2} - t\frac{dy}{dt} - n(n+2)y = 0$$

in terms of n.

3. (*40 points*) Give the complete solution of the wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

if $0 \le x \le 5$, with initial and boundary conditions

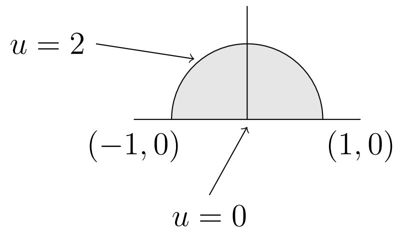
$$\begin{aligned} u(0,t) &= 0 \\ u(5,t) &= 0 \\ u(x,0) &= 0 \\ u_t(x,0) &= f(x) \end{aligned}$$

Be sure to explain carefully why you can eliminate various values of the separation constant λ.

4. (*20 points*) In polar coordinates, we have

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

Solve Laplace's equation $\nabla^2 u = 0$ in the semicircular region described by $0 \le r \le 1$, $0 \le \theta \le \pi$,



with boundary conditions $u(1, \theta) = 2$ for $0 < \theta < \pi$, $u(r, 0) = 0$ for $0 < r < 1$, and $u(r, \pi) = 0$ for $0 < r < 1$.