1. (20 points) The Helmholtz equation is

\[ \nabla^2 u = -ku. \]

In spherical coordinates,

\[ \nabla^2 u = \frac{\partial^2 u}{\partial \rho^2} + 2 \frac{\partial u}{\partial \rho} + \frac{1}{\rho^2} \left( \frac{\partial^2 u}{\partial \theta^2} + \frac{\cos \theta \partial u}{\sin \theta \partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2} \right). \]

Write \( u(\rho, \theta, \phi) = R(\rho)Y(\theta, \phi) \), and perform a separation of variables to get a partial differential equation for \( Y(\theta, \phi) \) and an ordinary differential equation for \( R(\rho) \). You should not try to solve either equation.

*Answer:* Write \( u(\rho, \theta, \phi) = R(\rho)Y(\theta, \phi) \), and we have

\[ R''Y + \frac{2R'}{\rho} - \frac{R}{\rho^2} \left( \frac{\partial^2 Y}{\partial \theta^2} + \frac{\cos \theta \partial Y}{\sin \theta \partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2 Y}{\partial \phi^2} \right) = -kR. \]

Multiply by \( \rho^2 \) and divide by \( RY \):

\[ \frac{\rho^2 R''}{R} + \frac{2\rho R'}{R} + \frac{1}{\rho^2} \left( \frac{\partial^2 Y}{\partial \theta^2} + \frac{\cos \theta \partial Y}{\sin \theta \partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2 Y}{\partial \phi^2} \right) = -k\rho^2. \]

Set both sides equal to \(-\lambda\):

\[ \frac{1}{\rho^2} \left( \frac{\partial^2 Y}{\partial \theta^2} + \frac{\cos \theta \partial Y}{\sin \theta \partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2 Y}{\partial \phi^2} \right) = -\lambda \]

\[ \frac{\partial^2 Y}{\partial \theta^2} + \frac{\cos \theta \partial Y}{\sin \theta \partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2 Y}{\partial \phi^2} + \lambda Y = 0 \quad (1) \]

\[ -k\rho^2 - \frac{\rho^2 R''}{R} - \frac{2\rho R'}{R} = -\lambda \]

\[ \rho^2 R'' + 2\rho R' + (k\rho^2 - \lambda)R = 0 \quad (2) \]

For the record, the partial differential equation (1) leads to what are called spherical harmonics, and the ordinary differential equation (2) leads to spherical Bessel functions.

2. (20 points) Let \( n \) be a positive integer. Give the general solution of

\[ t^2 \frac{d^2 y}{dt^2} - t \frac{dy}{dt} - n(n + 2)y = 0 \]

in terms of \( n \).

*Answer:* We write \( y = t^k \), and get \( k(k-1)t^{k-1} - kt^{k-1} - n(n+2)t^k = 0 \). Cancellation of \( t^k \) leads to \( k^2 - 2k - n(n+2) = 0 \). Factor to get \( (k+n)(k-(n+2)) = 0 \), with solutions \( k = n + 2 \) and \( k = -n \). The general solution is therefore \( y = At^{n+2} + Bt^{-k} \).
3. (40 points) Give the complete solution of the wave equation
\[
\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}
\]
if \(0 \leq x \leq 5\), with initial and boundary conditions
\[
\begin{align*}
  u(0, t) &= 0 \\
  u(5, t) &= 0 \\
  u(x, 0) &= 0 \\
  u_t(x, 0) &= f(x)
\end{align*}
\]
Be sure to explain carefully why you can eliminate various values of the separation constant \(\lambda\).

\textbf{Answer:} Separate variables, and we have \(u(x, t) = X(x)T(t)\). The differential equation becomes \(X''T = \frac{1}{c^2}XT''\), or \(\frac{X''}{X} = \frac{T''}{c^2 T} = -\lambda\). Now, \(1\) says that \(X(0) = 0\) and \(2\) says that \(X(5) = 0\).

Consider the differential equation \(X'' + \lambda X = 0\). If \(\lambda = -\alpha^2\), we have \(X = A \sinh \alpha x + B \cosh \alpha x\). The equation \(X(0) = 0\) forces \(B = 0\). Because \(\sinh x > 0\) for \(x > 0\), we now apply \(X(5) = 0\) to conclude that \(A = 0\). If \(\lambda = 0\), we have \(X'' = A + Bx\). Again, \(X(0) = 0\) tell us that \(A = 0\), and now \(X(5) = 0\) tells us that \(B = 0\).

We are left with \(\lambda = \alpha^2\), and so \(X'' + \alpha^2 X = 0\) has solution \(X = A \cos \alpha x + B \sin \alpha x\). Now \(X(0) = 0\) forces \(A = 0\), and \(X(5) = 0\) forces \(\sin 5\alpha = 0\). This has solution \(5\alpha = n\pi\) for \(n\) any positive integer, and so \(\alpha_n = n\pi/5\). We have \(T'' + c^2 \alpha_n^2 T = 0\) with solution \(T = A \cos c\alpha_n t + B \sin c\alpha_n t\).

Condition \(5\) forces \(T(0) = 0\), implying that \(A = 0\). We now have \(u_n(x, t) = A_n (\sin c\alpha_n t)(\sin \alpha_n x)\), and
\[
\begin{align*}
  u(x, t) &= \sum_{n=1}^{\infty} A_n \sin \frac{cn\pi t}{5} \sin \frac{n\pi x}{5} \\
  u_t(x, t) &= \sum_{n=1}^{\infty} A_n \frac{cn\pi}{5} \cos \frac{cn\pi t}{5} \sin \frac{n\pi x}{5} \\
  f(x) &= \sum_{n=1}^{\infty} A_n \frac{cn\pi}{5} \sin \frac{n\pi x}{5}
\end{align*}
\]
and the theory of Fourier series says that
\[
\begin{align*}
  A_n \frac{cn\pi}{5} &= \frac{2}{5} \int_0^5 f(x) \sin \frac{n\pi x}{5} \, dx \\
  A_n &= \frac{2}{c n \pi} \int_0^5 f(x) \sin \frac{n\pi x}{5} \, dx
\end{align*}
\]

4. (20 points) In polar coordinates, we have
\[
\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}.
\]
Solve Laplace’s equation \(\nabla^2 u = 0\) in the semicircular region described by \(0 \leq r \leq 1\), \(0 \leq \theta \leq \pi\),
with boundary conditions \( u(1, \theta) = 2 \) for \( 0 < \theta < \pi \), \( u(r, 0) = 0 \) for \( 0 < r < 1 \), and \( u(r, \pi) = 0 \) for \( 0 < r < 1 \).

**Answer:** We write \( u(r, \theta) = R(r)\Theta(\theta) \), and get

\[
R''\Theta + \frac{1}{r} R'\Theta + \frac{1}{r^2} R\Theta'' = 0
\]

Divide by \( R\Theta \) and multiply by \( r^2 \):

\[
\frac{r^2 R''}{R} + \frac{r R'}{R} + \frac{\Theta''}{\Theta} = 0.
\]

We have \( \Theta'' + k\Theta = 0 \). The boundary conditions require \( \Theta(0) = 0 \) and \( \Theta(\pi) = 0 \), and the same argument as in the previous problem forces \( k = n^2 \) for \( n \) a positive integer, and \( \Theta(\theta) = \sin n\theta \).

We now have \( r^2 R'' + r R' - n^2 R = 0 \). This is a Cauchy–Euler equation, with solution \( R = A r^n + B r^{-n} \). We require the function \( R(r) \) to be defined for \( r = 0 \), meaning that \( R_n(r) = a_n r^n \), \( u_n(r, \theta) = a_n r^n \sin n\theta \), and

\[
u(r, \theta) = \sum_{n=1}^{\infty} a_n r^n \sin n\theta
\]

The equation \( u(1, \theta) = 2 \) forces

\[
2 = \sum_{n=1}^{\infty} a_n \sin n\theta
\]

with solution

\[
a_n = \frac{2}{\pi} \int_{0}^{\pi} 2 \sin n\theta \, d\theta = \frac{4}{\pi} \left[ -\cos n\theta \right]_{0}^{\pi} = \frac{4}{\pi} \frac{1 - (-1)^n}{n}.
\]

We have

\[
u(r, \theta) = \sum_{n=1}^{\infty} \frac{4(1 - (-1)^n)}{\pi n} r^n \sin n\theta = \frac{8}{\pi} \sum_{k=0}^{\infty} \frac{r^{2k+1} \sin(2k+1)\theta}{2k + 1}.
\]
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Mean: 76.85
Standard deviation: 16.60