MT305.01: Advanced Calculus for Science Majors
Examination 4
Answers

1. (*20 points*) The *Helmholtz equation* is

$$\nabla^2 u = -ku.$$

In spherical coordinates,

$$\nabla^2 u = \frac{\partial^2 u}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\left(\frac{\partial^2 u}{\partial \theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial u}{\partial \theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2}\right).$$

Write $u(\rho, \theta, \phi) = R(\rho)Y(\theta, \phi)$, and perform a separation of variables to get a partial differential equation for $Y(\theta, \phi)$ and an ordinary differential equation for $R(\rho)$. *You should not try to solve either equation.*

Answer: Write $u(\rho, \theta, \phi) = R(\rho)Y(\theta, \phi)$, and we have

$$R''Y + \frac{2R'Y}{\rho} + \frac{R}{\rho^2}\left(\frac{\partial^2 Y}{\partial \theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial Y}{\partial \theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial \phi^2}\right) = -kRY.$$

Multiply by ρ^2 and divide by RY:

$$\frac{\rho^2 R''}{R} + \frac{2\rho R'}{R} + \frac{1}{Y}\left(\frac{\partial^2 Y}{\partial \theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial Y}{\partial \theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial \phi^2}\right) = -k\rho^2$$

$$\frac{1}{Y}\left(\frac{\partial^2 Y}{\partial \theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial Y}{\partial \theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial \phi^2}\right) = -k\rho^2 - \frac{\rho^2 R''}{R} - \frac{2\rho R'}{R}$$

Set both sides equal to $-\lambda$:

$$\frac{1}{Y}\left(\frac{\partial^2 Y}{\partial \theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial Y}{\partial \theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial \phi^2}\right) = -\lambda$$

$$\frac{\partial^2 Y}{\partial \theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial Y}{\partial \theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial \phi^2} + \lambda Y = 0 \tag{1}$$

$$-k\rho^2 - \frac{\rho^2 R''}{R} - \frac{2\rho R'}{R} = -\lambda$$

$$\rho^2 R'' + 2\rho R' + (k\rho^2 - \lambda)R = 0 \tag{2}$$

For the record, the partial differential equation (1) leads to what are called *spherical harmonics*, and the ordinary differential equation (2) leads to *spherical Bessel functions.*

2. (*20 points*) Let n be a positive integer. Give the general solution of

$$t^2\frac{d^2y}{dt^2} - t\frac{dy}{dt} - n(n+2)y = 0$$

in terms of n.

Answer: We write $y = t^k$, and get $k(k-1)t^k - kt^k - n(n+2)t^k = 0$. Cancellation of t^k leads to $k^2 - 2k - n(n+2) = 0$. Factor to get $(k+n)(k-(n+2) = 0$, with solutions $k = n+2$ and $k = -n$. The general solution is therefore $y = At^{n+2} + Bt^{-k}$.

3. (*40 points*) Give the complete solution of the wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

if $0 \le x \le 5$, with initial and boundary conditions

$$u(0,t) = 0 \tag{3}$$
$$u(5,t) = 0 \tag{4}$$
$$u(x,0) = 0 \tag{5}$$
$$u_t(x,0) = f(x) \tag{6}$$

Be sure to explain carefully why you can eliminate various values of the separation constant λ.

Answer: Separate variables, and we have $u(x,t) = X(x)T(t)$. The differential equation becomes $X''T = \frac{1}{c^2}XT''$, or $\dfrac{X''}{X} = \dfrac{T''}{c^2T} = -\lambda$. Now, (1) says that $X(0) = 0$ and (2) says that $X(5) = 0$.

Consider the differential equation $X'' + \lambda X = 0$. If $\lambda = -\alpha^2$, we have $X = A\sinh\alpha x + B\cosh\alpha x$. The equation $X(0) = 0$ forces $B = 0$. Because $\sinh x > 0$ for $x > 0$, we now apply $X(5) = 0$ to conclude that $A = 0$. If $\lambda = 0$, we have $X'' = A + Bx$. Again, $X(0) = 0$ tells us that $A = 0$, and now $X(5) = 0$ tells us that $B = 0$.

We are left with $\lambda = \alpha^2$, and so $X'' + \alpha^2 X = 0$ has solution $X = A\cos\alpha x + B\sin\alpha x$. Now $X(0) = 0$ forces $A = 0$, and $X(5) = 0$ forces $\sin 5\alpha = 0$. This has solution $5\alpha = n\pi$ for n any positive integer, and so $\alpha_n = n\pi/5$. We have $T'' + c^2\alpha_n^2 T = 0$ with solution $T = A\cos c\alpha_n t + B\sin c\alpha_n t$.

Condition (5) forces $T(0) = 0$, implying that $A = 0$. We now have $u_n(x,t) = A_n(\sin c\alpha_n t)(\sin\alpha_n x)$, and

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} A_n \sin\frac{cn\pi t}{5}\sin\frac{n\pi x}{5} \\ u_t(x,t) &= \sum_{n=1}^{\infty} A_n\frac{cn\pi}{5}\cos\frac{cn\pi t}{5}\sin\frac{n\pi x}{5} \\ f(x) &= \sum_{n=1}^{\infty} A_n\frac{cn\pi}{5}\sin\frac{n\pi x}{5} \end{aligned}$$

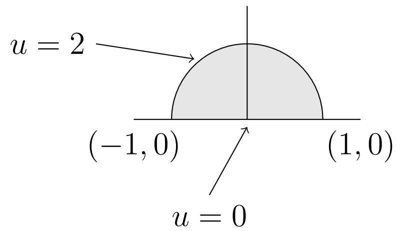
and the theory of Fourier series says that

$$\begin{aligned} A_n\frac{cn\pi}{5} &= \frac{2}{5}\int_0^5 f(x)\sin\frac{n\pi x}{5}\,dx \\ A_n &= \frac{2}{cn\pi}\int_0^5 f(x)\sin\frac{n\pi x}{5}\,dx \end{aligned}$$

4. (*20 points*) In polar coordinates, we have

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

Solve Laplace's equation $\nabla^2 u = 0$ in the semicircular region described by $0 \le r \le 1$, $0 \le \theta \le \pi$,



with boundary conditions $u(1,\theta) = 2$ for $0 < \theta < \pi$, $u(r, 0) = 0$ for $0 < r < 1$, and $u(r, \pi) = 0$ for $0 < r < 1$.

Answer: We write $u(r,\theta) = R(r)\Theta(\theta)$, and get

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0$$

Divide by $R\Theta$ and multiply by r^2:

$$\frac{r^2R''}{R} + \frac{rR'}{R} + \frac{\Theta''}{\Theta} = 0.$$
$$\frac{r^2R''}{R} + \frac{rR'}{R} = -\frac{\Theta''}{\Theta} = k.$$

We have $\Theta'' + k\Theta = 0$. The boundary conditions require $\Theta(0) = 0$ and $\Theta(\pi) = 0$, and the same argument as in the previous problem forces $k = n^2$ for n a positive integer, and $\Theta(\theta) = \sin n\theta$.

We now have $r^2R'' + rR' - n^2R = 0$. This is a Cauchy–Euler equation, with solution $R = Ar^n + Br^{-n}$. We require the function $R(r)$ to be defined for $r = 0$, meaning that $R_n(r) = a_nr^n$, $u_n(r,\theta) = a_nr^n \sin n\theta$, and

$$u(r,\theta) = \sum_{n=1}^{\infty} a_n r^n \sin n\theta$$

The equation $u(1,\theta) = 2$ forces

$$2 = \sum_{n=1}^{\infty} a_n \sin n\theta$$

with solution

$$a_n = \frac{2}{\pi}\int_0^{\pi} 2\sin n\theta \, d\theta = \frac{4}{\pi}\left[\frac{-\cos n\theta}{n}\right]_0^{\pi} = \frac{4}{\pi}\frac{1-(-1)^n}{n}.$$

We have

$$u(r,\theta) = \sum_{n=1}^{\infty} \frac{4(1-(-1)^n)}{\pi n} r^n \sin n\theta = \frac{8}{\pi}\sum_{k=0}^{\infty} \frac{r^{2k+1}\sin(2k+1)\theta}{2k+1}.$$

Grade	Number of people
100	1
94	1
90	2
89	1
85	1
84	1
82	1
67	1
58	1
55	1
53	1
52	1

Mean: 76.85
Standard deviation: 16.60