MT305.01: Advanced Calculus for Science Majors
Final Examination
Wednesday, May 9, 2012, 9 AM
Carney 206

Do all of your work in the blue booklets. Please label your answers clearly, as I will not have time to perform extensive searches for answers. No credit will be given for answers without explanations.

Cheating will result in a failing grade.

Calculators may not be used during this examination.

The problems are not arranged in order of increasing difficulty, so you might want to read all of them before beginning.

1. (*15 points*) Find the general solution to the differential equation

$$\frac{d^2y}{dx^2} + 4y = \sin(2x).$$

2. (*10 points*) Suppose that $f(t)$ is a differentiable function of exponential order, so that we can compute its Laplace transform. Suppose that $\mathscr{L}(f) = F(s)$. Derive the formula

$$\mathscr{L}(f'(t)) = sF(s) - f(0).$$

3. (*10 points*) Suppose that a and b are non-zero real numbers. Find the general solution of

$$\frac{dy}{dx} = ax + by.$$

4. (*15 points*) Solve the differential equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ for $0 \le x \le \pi$, $0 \le y \le \pi$ with the boundary conditions

$$\begin{aligned} u_x(0, y) &= u(0, y) \\ u(\pi, y) &= 2 \\ u(x, 0) &= 0 \\ u(x, \pi) &= 0 \end{aligned}$$

Be sure to explain fully how you arrived at the possible values of the separation constant.

5. (*10 points*) Write a solution to the differential equation

$$(t^2 + 2t)\frac{d^2y}{dt^2} + 2(t+1)\frac{dy}{dt} - 7y = 0$$

in the form $y = \sum_{n=0}^{\infty} a_n t^{n+r}$, with $a_0 = 1$. Show that $r = 0$, and compute the first 3 non-zero coefficients of the power series (not including a_0).

6. (*10 points*) Let b be a positive real number which is not an integer. Write $\cos bx$ in a Fourier series:

$$\cos bx = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx, \qquad -\pi \le x \le \pi$$

Compute a_0, a_n, and b_n in terms of b. Then substitute $x = \pi$ into the Fourier series and rearrange to get the formula

$$\pi b \cot \pi b = 1 + \sum_{n=1}^{\infty} \frac{2b^2}{b^2 - n^2}.$$

7. (*15 points*) Give the general solution of

$$\frac{d^2x}{dt^2} - \frac{dx}{dt} + \frac{x}{4} = \frac{\sqrt{te^t}}{4}.$$

8. (*15 points*) Solve the differential equation $\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} = \dfrac{\partial u}{\partial t}$ for $0 \leq r \leq a$ with boundary condition $u(a,t) = 0$ and initial conditions $u(r,0) = g(r)$. Be sure to explain fully how you arrived at the possible values of the separation constant.

Variation of Parameters

If $y_h = C_1y_1 + C_2y_2$ is the general solution to $y'' + P(x)y' + Q(x)y = 0$, and $y_p = u_1y_1 + u_2y_2 + y_h$ solves $y'' + P(x)y' + Q(x)y = f(x)$, then

$$\begin{aligned} y_1u_1' + y_2u_2' &= 0 \\ y_1'u_1' + y_2'u_2' &= f(x) \end{aligned}$$