MT305.01: Advanced Calculus for Science Majors
Final Examination
Answers

1. (*15 points*) Find the general solution to the differential equation

$$\frac{d^2y}{dx^2} + 4y = \sin(2x).$$

Answer: One way to proceed is to use undetermined coefficients. Variations of parameters also works, if you prefer that method. The solution of the homogeneous differential equation $y'' + 4y = 0$ is $y_h = C_1 \cos 2x + C_2 \sin 2x$. Therefore, the guess must be $y_p = Ax \cos 2x + Bx \sin 2x = x(A \cos 2x + B \sin 2x)$. We compute

$$\begin{aligned} y_p' &= A \cos 2x + B \sin 2x + x(-2A \sin 2x + 2B \cos 2x) \\ y_p'' &= -4A \sin 2x + 4B \cos 2x + x(-4A \cos 2x - 4B \sin 2x) \end{aligned}$$

and so

$$\begin{aligned} y_p'' + 4y_p &= -4A \sin 2x + 4B \cos 2x + x(-4A \cos 2x - 4B \sin 2x) + 4x(A \cos 2x + B \sin 2x) \\ &= -4A \sin 2x + 4B \cos 2x \end{aligned}$$

Setting $-4A \sin 2x + 4B \cos 2x$ equal to $\sin 2x$, we conclude that $B = 0$ and $-4A = 1$, implying $A = -1/4$.

Therefore, the general solution is $y = -x \cos(2x)/4 + C_1 \cos(2x) + C_2 \sin(2x)$.

2. (*10 points*) Suppose that $f(t)$ is a differentiable function of exponential order, so that we can compute its Laplace transform. Suppose that $\mathscr{L}(f) = F(s)$. Derive the formula

$$\mathscr{L}(f'(t)) = sF(s) - f(0).$$

Answer: We use the definition and integrate by parts, setting $u = e^{-st}$, $dv = f'(t)\,dt$, $du = -se^{-st}\,dt$, and $v = f(t)$:

$$\begin{aligned} \mathscr{L}(f'(t)) &= \int_0^\infty f'(t)e^{-st}\,dt = f(t)e^{-st}\Big|_0^\infty + \int_0^\infty f(t)se^{-st}\,dt \\ &= f(t)e^{-st}\Big|_0^\infty + s\int_0^\infty f(t)e^{-st}\,dt = f(t)e^{-st}\Big|_0^\infty + sF(s) \end{aligned}$$

Now, if $s > 0$, we know that $\lim_{t\to\infty} f(t)e^{-st} = 0$, because $f(t)$ has exponential order, so we conclude that $f(t)e^{-st}\Big|_0^\infty + sF(s) = -f(0) + sF(s)$.

3. (*10 points*) Suppose that a and b are non-zero real numbers. Find the general solution of

$$\frac{dy}{dx} = ax + by.$$

Answer: Rewrite the differential equation as $y' - by = ax$, and we see that an integrating factor is e^{-bx}. Multiply by this factor, and the equation becomes $axe^{-bx} = e^{-bx}y' - be^{-bx}y = (ye^{-bx})'$. Integration yields

$$ye^{-bx} = \int axe^{-bx}\,dx = -\frac{axe^{-bx}}{b} - \frac{ae^{-bx}}{b^2} + C$$

$$y = -\frac{ax}{b} - \frac{a}{b^2} + Ce^{bx}$$

4. (*15 points*) Solve the differential equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ for $0 \le x \le \pi$, $0 \le y \le \pi$ with the boundary conditions

(1) $$u_x(0, y) = u(0, y)$$
(2) $$u(\pi, y) = 2$$
(3) $$u(x, 0) = 0$$
(4) $$u(x, \pi) = 0$$

Be sure to explain fully how you arrived at the possible values of the separation constant.

Answer: We write $u(x, y) = X(x)Y(y)$, and arrive at the differential equation $X''Y + XY'' = 0$. Boundary condition (3) implies that $Y(0) = 0$, and boundary condition (4) implies that $Y(\pi) = 0$.

Separate the differential equation into $-X''/X = Y''/Y = -k$, so $Y'' + kY = 0$. We have the usual three possibilities:

(*i*) $k = 0$. In this case, $Y = ay + b$. The condition $Y(0) = 0$ forces $b = 0$, and then the condition $Y(\pi) = 0$ forces $a = 0$. We are left with the trivial solution.

(*ii*) $k = -\alpha^2 < 0$, with $\alpha > 0$. In this case, $Y = A \cosh \alpha y + B \sinh \alpha y$. The condition $Y(0) = 0$ forces $A = 0$ and $Y = B \sinh \alpha y$. The condition $Y(\pi) = 0$ forces $B = 0$, because $\sinh t > 0$ if $t > 0$.

(*iii*) $k = \alpha^2 > 0$, with $\alpha > 0$. In this case, $Y = A \cos \alpha y + B \sin \alpha y$. The condition $Y(0) = 0$ forces $A = 0$, and the condition $Y(\pi) = 0$ forces $\sin \alpha\pi = 0$, with conclusion $\alpha = n$, $k = n^2$, and $Y_n = \sin ny$.

We now confront $X''/X = n^2$, or $X'' - n^2 X = 0$, with solution $X_n = A \cosh nx + B \sinh nx$. Boundary condition (1) implies that $X'(0) = X(0)$. We see that $X(0) = A$ and $X'(0) = nB$, so $A = nB$. Substitution yields $X_n = B_n(n \cosh nx + \sinh nx)$. Therefore, $u_n(x, y) = B_n(n \cosh nx + \sinh nx) \sin ny$.

Finally, we apply boundary condition (2):

$$u(x, y) = \sum_{n=1}^{\infty} u_n(x, y) = \sum_{n=1}^{\infty} B_n(n \cosh nx + \sinh nx) \sin ny$$
$$2 = \sum_{n=1}^{\infty} B_n(n \cosh n\pi + \sinh n\pi) \sin ny$$

and the theory of Fourier series tells us that

$$B_n(n \cosh n\pi + \sinh n\pi) = \frac{2}{\pi} \int_0^{\pi} 2 \sin ny \, dy = \frac{4}{n\pi}(1 - \cos n\pi) = \frac{4(1 - (-1)^n)}{n\pi}$$
$$B_n = \frac{4(1 - (-1)^n)}{n\pi(n \cosh n\pi + \sinh n\pi)}$$
$$u(x, y) = \sum_{n=1}^{\infty} \frac{4(1 - (-1)^n)}{n\pi(n \cosh n\pi + \sinh n\pi)}(n \cosh nx + \sinh nx) \sin ny$$

5. (*10 points*) Write a solution to the differential equation

$$(t^2+2t)\frac{d^2y}{dt^2}+2(t+1)\frac{dy}{dt}-7y=0$$

in the form $y=\sum_{n=0}^{\infty} a_n t^{n+r}$, with $a_0=1$. Show that $r=0$, and compute the first 3 non-zero coefficients of the power series (not including a_0).

Answer: We have

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n t^{n+r} \\
\frac{dy}{dt} &= \sum_{n=0}^{\infty} a_n(n+r)t^{n+r-1} \\
2t\frac{dy}{dt} &= \sum_{n=0}^{\infty} 2a_n(n+r)t^{n+r} \\
\frac{d^2y}{dt^2} &= \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)t^{n+r-2} \\
2t\frac{d^2y}{dt^2} &= \sum_{n=0}^{\infty} 2a_n(n+r)(n+r-1)t^{n+r-1} \\
t^2\frac{d^2y}{dt^2} &= \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)t^{n+r}
\end{aligned}$$

and therefore

$$\begin{aligned}
t^2y''+2ty''+2ty'+2y'-7y &= \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)t^{n+r} \\
&\quad+\sum_{n=0}^{\infty} 2a_n(n+r)(n+r-1)t^{n+r-1} \\
&\quad+\sum_{n=0}^{\infty} 2a_n(n+r)t^{n+r}+\sum_{n=0}^{\infty} 2a_n(n+r)t^{n+r-1} \\
&\quad-\sum_{n=0}^{\infty} 7a_n t^{n+r}=0.
\end{aligned}$$

We start by computing the coefficient of t^{r-1}, which occurs only in the second and fourth sums when $n=0$. We have $(2a_0r(r-1)+2a_0r)t^{r-1}=0$. Because $a_0=1$, we get the equation $r^2=0$, with solution $r=0$.

Rewriting the equation with $r=0$, we have

$$\begin{aligned}
t^2y''+2ty''+2ty'+2y'-7y &= \sum_{n=0}^{\infty} a_n(n)(n-1)t^n+\sum_{n=0}^{\infty} 2a_n(n)(n-1)t^{n-1} \\
&\quad+\sum_{n=0}^{\infty} 2a_n(n)t^n+\sum_{n=0}^{\infty} 2a_n(n)t^{n-1}-\sum_{n=0}^{\infty} 7a_n t^n=0.
\end{aligned}$$

Reindex the second and fourth sums, and we have

$$\sum_{n=0}^{\infty} a_n(n)(n-1)t^n + \sum_{n=0}^{\infty} 2a_{n+1}(n+1)(n)t^n$$
$$+ \sum_{n=0}^{\infty} 2a_n(n)t^n + \sum_{n=0}^{\infty} 2a_{n+1}(n+1)t^n - \sum_{n=0}^{\infty} 7a_n t^n = 0$$
$$a_n(n^2+n-7) = -2a_{n+1}(n+1)^2$$
$$a_{n+1} = -\frac{a_n(n^2+n-7)}{2(n+1)^2}$$
$$a_0 = 1$$
$$a_1 = -\frac{-7}{2} = \frac{7}{2}$$
$$a_2 = -\frac{7}{2}\left(\frac{-5}{8}\right) = \frac{35}{16}$$
$$a_3 = -\frac{35}{16}\left(\frac{-1}{18}\right) = \frac{35}{288}$$

6. (*10 points*) Let b be a positive real number which is not an integer. Write $\cos bx$ in a Fourier series:

$$\cos bx = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx, \qquad -\pi \le x \le \pi$$

Compute a_0, a_n, and b_n in terms of b. Then substitute $x = \pi$ into the Fourier series and rearrange to get the formula

$$\pi b \cot \pi b = 1 + \sum_{n=1}^{\infty} \frac{2b^2}{b^2 - n^2}.$$

Answer: Because $\cos bx$ is an even function, we know immediately that $b_n = 0$. We compute

$$a_0 = \frac{1}{\pi}\int_0^{\pi} \cos bx\,dx = \frac{1}{b\pi}\sin bx\Big|_0^{\pi} = \frac{\sin b\pi}{b\pi}$$
$$a_n = \frac{2}{\pi}\int_0^{\pi} \cos nx \cos bx\,dx = \frac{2}{\pi(b^2-n^2)}(b\cos(nx)\sin(bx) - n\sin(nx)\cos(bx))\Big|_0^{\pi}$$
$$= \frac{2}{\pi(b^2-n^2)}(b\cos(n\pi)\sin(b\pi)) = \frac{2b(-1)^n\sin(b\pi)}{\pi(b^2-n^2)}$$
$$\cos bx = \frac{\sin b\pi}{b\pi} + \sum_{n=1}^{\infty}\frac{2b(-1)^n\sin(b\pi)}{\pi(b^2-n^2)}\cos nx = \frac{\sin b\pi}{b\pi}\left(1 + \sum_{n=1}^{\infty}\frac{2b^2(-1)^n\cos nx}{b^2-n^2}\right)$$

Substitute $x = \pi$:

$$\cos b\pi = \frac{\sin b\pi}{b\pi}\left(1 + \sum_{n=1}^{\infty}\frac{2b^2(-1)^n\cos n\pi}{b^2-n^2}\right) = \frac{\sin b\pi}{b\pi}\left(1 + \sum_{n=1}^{\infty}\frac{2b^2}{b^2-n^2}\right)$$
$$\frac{b\pi\cos b\pi}{\sin b\pi} = 1 + \sum_{n=1}^{\infty}\frac{2b^2}{b^2-n^2}$$

7. (*15 points*) Give the general solution of

$$\frac{d^2x}{dt^2} - \frac{dx}{dt} + \frac{x}{4} = \frac{\sqrt{te^t}}{4}.$$

Answer: We use variation of parameters. To solve $x'' - x' + x/4 = 0$, we solve $k^2 - k + 1/4 = 0$, with solution $k = 1/2$, and therefore the solution of the homogeneous differential equation is $x = Ae^{t/2} + Bte^{t/2}$. Write $y_1 = e^{t/2}$ and $y_2 = te^{t/2}$, with $y_1' = e^{t/2}/2$ and $y_2' = e^{t/2} + te^{t/2}/2$, and we have

$$\begin{aligned} u_1' e^{t/2} + u_2' te^{t/2} &= 0 \\ u_1' e^{t/2}/2 + u_2'(te^{t/2}/2 + e^{t/2}) &= \frac{t^{1/2}e^{t/2}}{4} \end{aligned}$$

Multiply the second equation by 2 to get:

$$u_1' e^{t/2} + u_2'(te^{t/2} + 2e^{t/2}) = \frac{t^{1/2}e^{t/2}}{2}$$

Subtract the first equation to get:

$$\begin{aligned} 2u_2' e^{t/2} &= \frac{t^{1/2}e^{t/2}}{2} \\ u_2' &= \frac{t^{1/2}}{4} \\ u_2 &= \frac{t^{3/2}}{6} \end{aligned}$$

Return to the first equation. Simplify and substitute:

$$\begin{aligned} u_1' e^{t/2} + u_2' te^{t/2} &= 0 \\ u_1' + u_2' t &= 0 \\ u_1' + \frac{t^{1/2}}{4} t &= 0 \\ u_1' &= -\frac{t^{3/2}}{4} \\ u_1 &= -\frac{t^{5/2}}{10} \\ y &= u_1 y_1 + u_2 y_2 + Ay_1 + By_2 \\ &= -\frac{t^{5/2}}{10} e^{t/2} + \frac{t^{3/2}}{6} te^{t/2} + Ae^{t/2} + Bte^{t/2} \\ &= \frac{t^{5/2}e^{t/2}}{15} + Ae^{t/2} + Bte^{t/2} \end{aligned}$$

8. (*15 points*) Solve the differential equation $\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} = \dfrac{\partial u}{\partial t}$ for $0 \le r \le a$ with boundary condition $u(a, t) = 0$ and initial conditions $u(r, 0) = g(r)$. Be sure to explain fully how you arrived at the possible values of the separation constant.

Answer: Write $u(r,t) = R(r)T(t)$, and the partial differential equation becomes $R''T + R'T/r = RT'$. Divide by RT, and we have $R''/R + R'/(rR) = T'/T = -k$. We have two linked differential equations: $rR'' + R' + krR = 0$ and $T' + kT = 0$.

We begin with the equation for $R(r)$, which we write $r^2R'' + rR' + kr^2R = 0$. The initial condition $u(a,t) = 0$ means that $R(a) = 0$. We have the usual three possibilities:

(*i*) $k = 0$. The solution of $rR'' + R' = 0$ is $R(r) = C + D\log r$. We require $R(0)$ to be defined, forcing $D = 0$, and $R(a) = 0$ forces $C = 0$.

(*ii*) $k = -\alpha^2 < 0$ with $\alpha > 0$. The solution of $r^2R'' + rR' - \alpha^2r^2R = 0$ is $R(r) = c_1I_0(\alpha r) + c_2K_0(\alpha r)$. We require $R(0)$ to be defined, forcing c_2 to be 0. We require $R(a) = 0$, and because $I_0(r) > 0$ if $r > 0$, we have $c_1 = 0$.

(*iii*) $k = \alpha^2 > 0$ with $\alpha > 0$. The solution of $r^2R'' + rR' + \alpha^2r^2R = 0$ is $R(r) = c_1J_0(\alpha r) + c_2Y_0(\alpha r)$. We require $R(0)$ to be defined, forcing c_2 to be 0. We require $R(a) = 0$, forcing $J_0(\alpha a) = 0$. We have a sequence of values, $0 < \alpha_1 < \alpha_2 < \alpha_3 < \cdots$, with $R_n(\alpha_n a) = 0$, and $k = \alpha_n^2$.

We now consider $T' + \alpha_n^2T = 0$, with solution $T_n = C_ne^{-\alpha_n^2t}$. We have $u_n(r,t) = R_n(r)T_n(t) = C_ne^{-\alpha_n^2t}J_0(\alpha_nr)$, and

$$u(r,t) = \sum_{n=1}^{\infty} C_ne^{-\alpha_n^2t}J_0(\alpha_nr).$$

Substitute $t = 0$, and we have

$$g(r) = \sum_{n=1}^{\infty} C_nJ_0(\alpha_nr).$$

The orthogonality relation for Bessel functions tells us that

$$C_n = \frac{\displaystyle\int_0^a J_0(\alpha_nr)g(r)r\,dr}{\displaystyle\int_0^a J_0^2(\alpha_nr)r\,dr} = \frac{2}{J_1^2(\alpha_na)}\int_0^a J_0(\alpha_nr)g(r)r\,dr$$

Grade	Number of people
90	1
85	1
76	1
73	1
70	1
64	1
60	2
48	1
47	1
45	1
35	1
31	1

Mean: 60.31
Standard deviation: 17.69