1. \(15\) \textit{points} Find the general solution to the differential equation
\[
\frac{d^2y}{dx^2} + 4y = \sin(2x).
\]
\textbf{Answer:} One way to proceed is to use undetermined coefficients. Variations of parameters also works, if you prefer that method. The solution of the homogeneous differential equation \(y'' + 4y = 0\) is \(y_h = C_1 \cos 2x + C_2 \sin 2x\). Therefore, the guess must be \(y_p = Ax \cos 2x + Bx \sin 2x = x(A \cos 2x + B \sin 2x)\). We compute
\[
\begin{align*}
y'_p &= A \cos 2x + B \sin 2x + x(-2A \sin 2x + 2B \cos 2x) \\
y''_p &= -4A \sin 2x + 4B \cos 2x + x(-4A \cos 2x - 4B \sin 2x)
\end{align*}
\]
and so
\[
y''_p + 4y_p = -4A \sin 2x + 4B \cos 2x + x(-4A \cos 2x - 4B \sin 2x) + 4x(A \cos 2x + B \sin 2x)
\]
\[
= -4A \sin 2x + 4B \cos 2x
\]
Setting \(-4A \sin 2x + 4B \cos 2x\) equal to \(\sin 2x\), we conclude that \(B = 0\) and \(-4A = 1\), implying \(A = -1/4\).
Therefore, the general solution is \(y = -x \cos(2x)/4 + C_1 \cos(2x) + C_2 \sin(2x)\).

2. \(10\) \textit{points} Suppose that \(f(t)\) is a differentiable function of exponential order, so that we can compute its Laplace transform. Suppose that \(\mathcal{L}(f) = F(s)\). Derive the formula
\[
\mathcal{L}(f'(t)) = sF(s) - f(0).
\]
\textbf{Answer:} We use the definition and integrate by parts, setting \(u = e^{-st}\), \(dv = f'(t)\,dt\), \(du = -se^{-st}\,dt\), and \(v = f(t)\):
\[
\mathcal{L}(f'(t)) = \int_0^\infty f'(t)e^{-st}\,dt = f(t)e^{-st}\bigg|_0^\infty + \int_0^\infty f(t)se^{-st}\,dt
\]
\[
= f(t)e^{-st}\bigg|_0^\infty + s \int_0^\infty f(t)e^{-st}\,dt = f(t)e^{-st}\bigg|_0^\infty + sF(s)
\]
Now, if \(s > 0\), we know that \(\lim_{t \to \infty} f(t)e^{-st} = 0\), because \(f(t)\) has exponential order, so we conclude that \(f(t)e^{-st}\bigg|_0^\infty + sF(s) = -f(0) + sF(s)\).

3. \(10\) \textit{points} Suppose that \(a\) and \(b\) are non-zero real numbers. Find the general solution of
\[
\frac{dy}{dx} = ax + by.
\]
\textbf{Answer:} Rewrite the differential equation as \(y' - by = ax\), and we see that an integrating factor is \(e^{-bx}\). Multiply by this factor, and the equation becomes \(axe^{-bx} = e^{-bx}y' - be^{-bx}y = (ye^{-bx})'\). Integration yields
\[
ye^{-bx} = \int axe^{-bx}\,dx = -\frac{axe^{-bx}}{b} - \frac{ae^{-bx}}{b^2} + C
\]
\[ y = -\frac{ax}{b} - \frac{a}{b^2} + Ce^{bx} \]

4. (15 points) Solve the differential equation \( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \) for \( 0 \leq x \leq \pi, 0 \leq y \leq \pi \) with the boundary conditions

(1) \( u_x(0, y) = u(0, y) \)
(2) \( u(\pi, y) = 2 \)
(3) \( u(x, 0) = 0 \)
(4) \( u(x, \pi) = 0 \)

Be sure to explain fully how you arrived at the possible values of the separation constant.

Answer: We write \( u(x, y) = X(x)Y(y) \), and arrive at the differential equation \( X''Y + XY'' = 0 \). Boundary condition (3) implies that \( Y(0) = 0 \), and boundary condition (4) implies that \( Y(\pi) = 0 \).

Separate the differential equation into \(-X''/X = Y''/Y = -k\), so \( Y'' + kY = 0 \). We have the usual three possibilities:

(i) \( k = 0 \). In this case, \( Y = ay + b \). The condition \( Y(0) = 0 \) forces \( b = 0 \), and then the condition \( Y(\pi) = 0 \) forces \( a = 0 \). We are left with the trivial solution.

(ii) \( k = -\alpha^2 < 0 \), with \( \alpha > 0 \). In this case, \( Y = A \cosh \alpha y + B \sinh \alpha y \). The condition \( Y(0) = 0 \) forces \( A = 0 \) and \( Y = B \sinh \alpha y \). The condition \( Y(\pi) = 0 \) forces \( B = 0 \), because \( \sinh t > 0 \) if \( t > 0 \).

(iii) \( k = \alpha^2 > 0 \), with \( \alpha > 0 \). In this case, \( Y = A \cos \alpha y + B \sin \alpha y \). The condition \( Y(0) = 0 \) forces \( A = 0 \), and the condition \( Y(\pi) = 0 \) forces \( \sin \alpha \pi = 0 \), with conclusion \( \alpha = n, k = n^2 \), and \( Y_n = \sin ny \).

We now confront \( X''/X = n^2 \), or \( X'' - n^2 X = 0 \), with solution \( X_n = A \cosh nx + B \sinh nx \). Boundary condition (1) implies that \( X'(0) = X(0) \). We see that \( X(0) = A \) and \( X'(0) = nB \), so \( A = nB \). Substitution yields \( X_n = B_n (n \cosh nx + \sinh nx) \). Therefore, \( u_n(x, y) = B_n (n \cosh nx + \sinh nx) \sin ny \).

Finally, we apply boundary condition (2):

\[
\begin{align*}
  u(x, y) &= \sum_{n=1}^{\infty} u_n(x, y) = \sum_{n=1}^{\infty} B_n (n \cosh nx + \sinh nx) \sin ny \\
  2 &= \sum_{n=1}^{\infty} B_n (n \cosh n\pi + \sinh n\pi) \sin ny
\end{align*}
\]

and the theory of Fourier series tells us that

\[
\begin{align*}
  B_n (n \cosh n\pi + \sinh n\pi) &= \frac{2}{\pi} \int_{0}^{\pi} 2 \sin ny \, dy = \frac{4}{n\pi} (1 - \cos n\pi) = \frac{4(1 - (-1)^n)}{n\pi} \\
  B_n &= \frac{4(1 - (-1)^n)}{n\pi(n \cosh n\pi + \sinh n\pi)} \\
  u(x, y) &= \sum_{n=1}^{\infty} \frac{4(1 - (-1)^n)}{n\pi(n \cosh n\pi + \sinh n\pi)} (n \cosh nx + \sinh nx) \sin ny
\end{align*}
\]
5. (10 points) Write a solution to the differential equation

\[(t^2 + 2t) \frac{d^2 y}{dt^2} + 2(t+1) \frac{dy}{dt} - 7y = 0\]

in the form \(y = \sum_{n=0}^{\infty} a_n t^{n+r}\), with \(a_0 = 1\). Show that \(r = 0\), and compute the first 3 non-zero coefficients of the power series (not including \(a_0\)).

Answer: We have

\[y = \sum_{n=0}^{\infty} a_n t^{n+r}\]
\[\frac{dy}{dt} = \sum_{n=0}^{\infty} a_n (n+r) t^{n+r-1}\]
\[2t \frac{dy}{dt} = \sum_{n=0}^{\infty} 2a_n (n + r) t^{n+r}\]
\[\frac{d^2 y}{dt^2} = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1)t^{n+r-2}\]
\[2t \frac{d^2 y}{dt^2} = \sum_{n=0}^{\infty} 2a_n (n + r)(n + r - 1)t^{n+r-1}\]
\[t^2 \frac{d^2 y}{dt^2} = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1)t^{n+r}\]

and therefore

\[t^2 y'' + 2ty'' + 2ty' + 2y' - 7y = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1)t^{n+r}\]
\[+ \sum_{n=0}^{\infty} 2a_n (n + r)(n + r - 1)t^{n+r-1}\]
\[+ \sum_{n=0}^{\infty} 2a_n (n + r)t^{n+r} + \sum_{n=0}^{\infty} 2a_n (n + r)t^{n+r-1}\]
\[- \sum_{n=0}^{\infty} 7a_n t^{n+r} = 0.\]

We start by computing the coefficient of \(t^{-1}\), which occurs only in the second and fourth sums when \(n = 0\). We have \((2a_0 r(r-1) + 2a_0 r)t^{-1} = 0\). Because \(a_0 = 1\), we get the equation \(r^2 = 0\), with solution \(r = 0\).

Rewriting the equation with \(r = 0\), we have

\[t^2 y'' + 2ty'' + 2ty' + 2y' - 7y = \sum_{n=0}^{\infty} a_n (n)(n-1)t^n + \sum_{n=0}^{\infty} 2a_n (n)(n-1)t^{n-1}\]
\[+ \sum_{n=0}^{\infty} 2a_n(n)t^n + \sum_{n=0}^{\infty} 2a_n(n)t^{n-1} - \sum_{n=0}^{\infty} 7a_n t^n = 0.\]
Reindex the second and fourth sums, and we have

\[
\begin{align*}
    \sum_{n=0}^{\infty} a_n (n-1)t^n + \sum_{n=0}^{\infty} 2a_{n+1} (n+1)(n)t^n \\
    + \sum_{n=0}^{\infty} 2a_n (n)t^n + \sum_{n=0}^{\infty} 2a_{n+1} (n+1)t^n - \sum_{n=0}^{\infty} 7a_n t^n = 0
\end{align*}
\]

\[
a_n(n^2 + n - 7) = -2a_{n+1}(n+1)^2
\]

\[
a_{n+1} = -\frac{a_n(n^2 + n - 7)}{2(n+1)^2}
\]

\[
a_0 = 1
\]

\[
a_1 = -\frac{7}{2} = \frac{7}{2}
\]

\[
a_2 = -\frac{7}{2} \left( \frac{-5}{8} \right) = \frac{35}{16}
\]

\[
a_3 = -\frac{35}{16} \left( \frac{-1}{18} \right) = \frac{35}{288}
\]

6. (10 points) Let \( b \) be a positive real number which is not an integer. Write \( \cos bx \) in a Fourier series:

\[
\cos bx = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx, \quad -\pi \leq x \leq \pi
\]

Compute \( a_0, a_n, \) and \( b_n \) in terms of \( b \). Then substitute \( x = \pi \) into the Fourier series and rearrange to get the formula

\[
\pi b \cot \pi b = 1 + \sum_{n=1}^{\infty} \frac{2b^2}{b^2 - n^2}.
\]

**Answer:** Because \( \cos bx \) is an even function, we know immediately that \( b_n = 0 \). We compute

\[
a_0 = \frac{1}{\pi} \int_{0}^{\pi} \cos bx \, dx = \frac{1}{b\pi} \sin bx \bigg|_{0}^{\pi} = \frac{\sin b\pi}{b\pi}
\]

\[
a_n = \frac{2}{\pi} \int_{0}^{\pi} \cos nx \cos bx \, dx = \frac{2}{\pi(b^2 - n^2)} (b\cos(n\pi) \sin(bx) - n\sin(nx) \cos(bx)) \bigg|_{0}^{\pi}
\]

\[
= \frac{2}{\pi(b^2 - n^2)} (b\cos(n\pi) \sin(b\pi)) = \frac{2b(-1)^n \sin(b\pi)}{\pi(b^2 - n^2)}
\]

\[
\cos bx = \frac{\sin b\pi}{b\pi} + \sum_{n=1}^{\infty} \frac{2b(-1)^n \sin(b\pi)}{\pi(b^2 - n^2)} \cos nx = \frac{\sin b\pi}{b\pi} \left( 1 + \sum_{n=1}^{\infty} \frac{2b^2(-1)^n \cos nx}{b^2 - n^2} \right)
\]

Substitute \( x = \pi \):

\[
\cos b\pi = \frac{\sin b\pi}{b\pi} \left( 1 + \sum_{n=1}^{\infty} \frac{2b^2(-1)^n \cos n\pi}{b^2 - n^2} \right) = \frac{\sin b\pi}{b\pi} \left( 1 + \sum_{n=1}^{\infty} \frac{2b^2}{b^2 - n^2} \right)
\]

\[
\frac{b\pi \cos b\pi}{\sin b\pi} = 1 + \sum_{n=1}^{\infty} \frac{2b^2}{b^2 - n^2}
\]
7. (15 points) Give the general solution of 
\[ \frac{d^2 x}{dt^2} - \frac{dx}{dt} + \frac{x}{4} = \sqrt{t} e^t. \]

Answer: We use variation of parameters. To solve \( x'' - x' + x/4 = 0 \), we solve \( k^2 - k + 1/4 = 0 \), with solution \( k = 1/2 \), and therefore the solution of the homogeneous differential equation is \( x = Ae^{t/2} + Bte^{t/2} \). Write \( y_1 = e^{t/2} \) and \( y_2 = te^{t/2} \), with \( y_1' = e^{t/2}/2 \) and \( y_2' = e^{t/2} + te^{t/2}/2 \), and we have

\[ u_1' e^{t/2} + u_2' te^{t/2} = 0 \]

\[ u_1' e^{t/2}/2 + u_2' (te^{t/2}/2 + e^{t/2}) = \frac{t^{1/2} e^{t/2}}{4} \]

Multiply the second equation by 2 to get:

\[ u_1' e^{t/2} + u_2' (te^{t/2}/2 + 2e^{t/2}) = \frac{t^{1/2} e^{t/2}}{2} \]

Subtract the first equation to get:

\[ 2u_2' e^{t/2} = \frac{t^{1/2} e^{t/2}}{2} \]

\[ u_2' = \frac{t^{1/2}}{4} \]

\[ u_2 = \frac{t^{3/2}}{6} \]

Return to the first equation. Simplify and substitute:

\[ u_1' e^{t/2} + u_2' te^{t/2} = 0 \]

\[ u_1' + u_2' t = 0 \]

\[ u_1' + \frac{t^{1/2}}{4} t = 0 \]

\[ u_1' = -\frac{t^{3/2}}{4} \]

\[ u_1 = -\frac{t^{5/2}}{10} \]

\[ y = u_1 y_1 + u_2 y_2 + Ay_1 + By_2 \]

\[ = -\frac{t^{5/2}}{10} e^{t/2} + \frac{t^{3/2}}{6} te^{t/2} + Ae^{t/2} + Bte^{t/2} \]

\[ = \frac{t^{5/2} e^{t/2}}{15} + Ae^{t/2} + Bte^{t/2} \]

8. (15 points) Solve the differential equation \( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} = \frac{\partial u}{\partial t} \) for \( 0 \leq r \leq a \) with boundary condition \( u(a, t) = 0 \) and initial conditions \( u(r, 0) = g(r) \). Be sure to explain fully how you arrived at the possible values of the separation constant.
Answer: Write \( u(r, t) = R(r)T(t) \), and the partial differential equation becomes \( R'' + \frac{R''}{R} + \frac{rR'}{rR} = T' + kT = 0 \). We have two linked differential equations: \( rR'' + R' + krR = 0 \) and \( T' + kT = 0 \).

We begin with the equation for \( R(r) \), which we write \( r^2 R'' + rR' + kr^2 R = 0 \). The initial condition \( u(a, t) = 0 \) means that \( R(a) = 0 \). We have the usual three possibilities:

(i) \( k = 0 \). The solution of \( rR'' + R' = 0 \) is \( R(r) = C + D \log r \). We require \( R(0) \) to be defined, forcing \( D = 0 \), and \( R(a) = 0 \) forces \( C = 0 \).

(ii) \( k = -\alpha^2 < 0 \) with \( \alpha > 0 \). The solution of \( r^2 R'' + rR' - \alpha^2 r^2 R = 0 \) is \( R(r) = c_1 I_0(\alpha r) + c_2 K_0(\alpha r) \). We require \( R(0) \) to be defined, forcing \( c_2 \) to be 0. We require \( R(a) = 0 \), and because \( I_0(r) > 0 \) if \( r > 0 \), we have \( c_1 = 0 \).

(iii) \( k = \alpha^2 > 0 \) with \( \alpha > 0 \). The solution of \( r^2 R'' + rR' + \alpha^2 r^2 R = 0 \) is \( R(r) = c_1 J_0(\alpha r) + c_2 Y_0(\alpha r) \). We require \( R(0) \) to be defined, forcing \( c_2 \) to be 0. We require \( R(a) = 0 \), forcing \( J_0(\alpha a) = 0 \). We have a sequence of values, \( 0 < \alpha_1 < \alpha_2 < \alpha_3 < \cdots \), with \( R_n(\alpha_n a) = 0 \), and \( k = \alpha_n^2 \).

We now consider \( T' + \alpha^2 nT = 0 \), with solution \( T_n = C_n e^{-\alpha_n^2 t} \). We have \( u_n(r, t) = R_n(r)T_n(t) = C_n e^{-\alpha_n^2 t} J_0(\alpha_n r) \), and

\[
u(r, t) = \sum_{n=1}^{\infty} C_n e^{-\alpha_n^2 t} J_0(\alpha_n r) .\]

Substitute \( t = 0 \), and we have

\[ g(r) = \sum_{n=1}^{\infty} C_n J_0(\alpha_n r) .\]

The orthogonality relation for Bessel functions tells us that

\[
C_n = \frac{\int_0^a J_0(\alpha_n r) g(r) r \, dr}{\int_0^a J_0^2(\alpha_n r) r \, dr} = \frac{2}{J_1(\alpha_n a)^2} \int_0^a J_0(\alpha_n r) g(r) r \, dr
\]

Grade | Number of people
--- | ---
90   | 1
85   | 1
76   | 1
73   | 1
70   | 1
64   | 1
60   | 2
48   | 1
47   | 1
45   | 1
35   | 1
31   | 1

Mean: 60.31
Standard deviation: 17.69