Mathematics 310
Examination 1
October 7, 2011

Please do all of your work in the blue booklets. Please work clearly and neatly, and label your answers. You do not need to do the problems in order. No credit will be given for answers without explanations.

Cheating will result in a failing grade.

No calculators may be used during this examination.

The problems are not arranged in order of increasing difficulty, so you might want to read all of them before beginning.

1. (*10 points*) Let G be a group, and let x be an element of G. Finish the following definition: The *order* of x is ...

2. (*10 points*) State Lagrange's Theorem.

3. (*10 points*) Let

$$H = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in \mathbf{Z}, ab \neq 0 \right\}.$$

Is H a group with the binary operation of matrix multiplication? Be sure to explain your answer fully.

4. (*20 points*) Suppose that G_1 and G_2 are groups, and $\phi : G_1 \to G_2$ is a homomorphism.
   (*a*) Recall that we defined $\phi(G_1) = \{\phi(g_1) : g_1 \in G_1\}$. Show that $\phi(G_1)$ is a subgroup of G_2.
   (*b*) Suppose that H_2 is a subgroup of G_2. Recall that we defined $\phi^{-1}(H_2) = \{g_1 \in G_1 : \phi(g_1) \in H_2\}$. Prove that $\phi^{-1}(H_2)$ is a subgroup of G_1.

5. (*10 points*) Suppose that $\phi : G_1 \to G_2$ is a homomorphism of finite groups. Suppose that $a \in G_1$. Prove that $o(\phi(a)) | o(a)$.

6. (*10 points*) Remember that $\mathrm{GL}_2(\mathbf{R})$ is defined to be the set of invertible 2-by-2 matrices with real entries, and $\mathrm{SL}_2(\mathbf{R})$ is the subgroup of $\mathrm{GL}_2(\mathbf{R})$ containing matrices with determinant 1. (You may assume that without proof that both $\mathrm{GL}_2(\mathbf{R})$ and $\mathrm{SL}_2(\mathbf{R})$ are groups.) Prove that $\mathrm{SL}_2(\mathbf{R})$ is a normal subgroup of $\mathrm{GL}_2(\mathbf{R})$.

7. (*30 points*) Let G be a finite group containing n elements.

Prove or give a counterexample to each of the following statements. Providing a *counterexample* means that you will tell me a specific group G and a specific explanation of why the group G violates the given statement.
   (*a*) Every element in G except for the identity has order n.
   (*b*) There must be *some* element in G which has order n.
   (*c*) If G is a finite *abelian* group containing n elements, then there must be *some* element in G which has order n.