Mathematics 310
Examination 1
Answers

1. (*10 points*) Let G be a group, and let x be an element of G. Finish the following definition: The *order* of x is ...

Answer: ...the smallest positive integer n so that $x^n = e$.

2. (*10 points*) State Lagrange's Theorem.

Answer: If G is a finite group, and H is a subgroup of G, then $o(H)|o(G)$.

3. (*10 points*) Let

$$H = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in \mathbf{Z}, ab \neq 0 \right\}.$$

Is H a group with the binary operation of matrix multiplication? Be sure to explain your answer fully.

Answer: This is not a group. The inverse of the matrix $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ is $\begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$, which is not in H.

4. (*20 points*) Suppose that G_1 and G_2 are groups, and $\phi : G_1 \to G_2$ is a homomorphism.

(*a*) Recall that we defined $\phi(G_1) = \{\phi(g_1) : g_1 \in G_1\}$. Show that $\phi(G_1)$ is a subgroup of G_2.

(*b*) Suppose that H_2 is a subgroup of G_2. Recall that we defined $\phi^{-1}(H_2) = \{g_1 \in G_1 : \phi(g_1) \in H_2\}$. Prove that $\phi^{-1}(H_2)$ is a subgroup of G_1.

Answer: (*a*) Pick $x, y \in \phi(G_1)$. Then we can write $x = \phi(a)$ and $y = \phi(b)$, with $a, b \in G_1$. Because G_1 is closed under the group operation, we know that $ab \in G_1$. Because ϕ is a homomorphism, we know that $xy = \phi(a)\phi(b) = \phi(ab)$, and therefore $xy \in \phi(G_1)$. That shows that $\phi(G_1)$ is closed under the group operation.

Because $\phi(e_1) = e_2$, and $e_1 \in G_1$, we know that $e_2 \in \phi(G_1)$.

Finally, if $x \in \phi(G_1)$, we write $x = \phi(a)$ with $a \in G_1$. Then $a^{-1} \in G_1$, and $\phi(a^{-1}) = \phi(a)^{-1} = x^{-1} \in \phi(G_1)$, showing that the inverse of each element in $\phi(G_1)$ is also in $\phi(G_1)$.

(*b*) Suppose that $a, b \in \phi^{-1}(H_2)$. Then $\phi(a), \phi(b) \in H_2$. Because H_2 is a subgroup, we know that $\phi(a)\phi(b) \in H_2$. Because ϕ is a homomorphism, we know that $\phi(a)\phi(b) = \phi(ab)$. Because $\phi(ab) \in H_2$, we know that $ab \in \phi^{-1}(H_2)$, so $\phi^{-1}(H_2)$ is closed under the group operation. Because $\phi(e_1) = e_2 \in H_2$, we know that $e_1 \in \phi^{-1}(H_2)$. Finally, if $a \in \phi^{-1}(H_2)$, we have $\phi(a) \in H_2$. Because the inverse of each element in H_2 is in H_2, we have $\phi(a)^{-1} \in H_2$. This is the same as $\phi(a^{-1}) \in H_2$, and that tells us that $a^{-1} \in \phi^{-1}(H_2)$, so the inverse of each element in $\phi^{-1}(H_2)$ is in $\phi^{-1}(H_2)$. This shows that $\phi^{-1}(H_2)$ is a subgroup.

5. (*10 points*) Suppose that $\phi : G_1 \to G_2$ is a homomorphism of finite groups. Suppose that $a \in G_1$. Prove that $o(\phi(a))|o(a)$.

Answer: Let $n = o(a)$. We know that $a^n = e_1$, and therefore $\phi(a^n) = e_2$. This says that $\phi(a)^n = e^2$. We proved in class that if $g^k = e$, then $o(g)|k$. Therefore, $o(\phi(a))|n$.

6. (*10 points*) Remember that $\mathrm{GL}_2(\mathbf{R})$ is defined to be the set of invertible 2-by-2 matrices with real entries, and $\mathrm{SL}_2(\mathbf{R})$ is the subgroup of $\mathrm{GL}_2(\mathbf{R})$ containing matrices with determinant

1. (You may assume that without proof that both $GL_2(\mathbf{R})$ and $SL_2(\mathbf{R})$ are groups.) Prove that $SL_2(\mathbf{R})$ is a normal subgroup of $GL_2(\mathbf{R})$.

Answer: Suppose that $A \in SL_2(\mathbf{R})$ and $B \in GL_2(\mathbf{R})$. We must show that $BAB^{-1} \in SL_2(\mathbf{R})$. Because $\det(BAB^{-1}) = \det(B)\det(A)\det(B^{-1}) = \det(B) \cdot 1 \cdot \det(B)^{-1} = 1$, we know that BAB^{-1} has determinant 1 and hence is in $SL_2(\mathbf{R})$.

7. (*30 points*) Let G be a finite group containing n elements.

Prove or give a counterexample to each of the following statements. Providing a *counterexample* means that you will tell me a specific group G and a specific explanation of why the group G violates the given statement.

(a) Every element in G except for the identity has order n.

(b) There must be *some* element in G which has order n.

(c) If G is a finite *abelian* group containing n elements, then there must be *some* element in G which has order n.

Answer: (a) This statement is *false.* The easiest counterexample is given by $\mathbf{Z}/4\mathbf{Z}$, which has 4 elements, and the element 2 has order 2, not order 4.

(b) This statement is *false.* One counterexample is to let $G = S_3$, which has no elements of order 6. In S_3, there is the identity, with order 1, three elements of order 2, and two elements of order 3. There cannot be an element of order 6, or else S_3 would be cyclic, and hence abelian.

(c) This statement is *false.* One counterexample is given by the group $\mathbf{Z}/8\mathbf{Z}^\times = U_8 = \{1, 3, 5, 7\}$, with group operation multiplication modulo 8. Each of the three non-identity elements has order 2, but the group has order order 4.

Grade	Number of people
70	1
69	1
65	2
62	2
45	2
37	1

Mean: 57.78

Standard deviation: 11.42