Mathematics 310
Examination 2
Answers

1. (*20 points*) Find the order of each of these elements of S_9, and identify each permutation as odd or even:

(a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 8 & 6 & 7 & 4 & 3 & 2 & 5 & 1 & 9 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 3 & 6 & 7 & 8 & 5 & 4 & 9 & 1 \end{pmatrix}$

Answer: (a) We have $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 8 & 6 & 7 & 4 & 3 & 2 & 5 & 1 & 9 \end{pmatrix} = (18)(26)(375) = (18)(26)(35)(37)$. The cycle decomposition shows that the permutation has order 6, and the transposition decomposition shows that the permutation is even.

(b) We have $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 3 & 6 & 7 & 8 & 5 & 4 & 9 & 1 \end{pmatrix} = (1236589)(47) = (19)(18)(15)(16)(13)(12)(47)$. The cycle decomposition shows that the permutation has order 14, and the transposition decomposition shows that the permutation is odd.

2. (*20 points*) Suppose that G_1 and G_2 are groups, and $\varphi : G_1 \to G_2$ is a group homomorphism. Prove or give a counterexample to each of these two statements:

(i) If G_2 is abelian and φ is surjective, then G_1 is abelian.

(ii) If G_1 is abelian and φ is injective, then G_2 is abelian.

A counterexample requires you to find specific groups G_1 and G_2, as well as the homomorphism φ.

Answer: (i) This statement is *false*. One example is to take $G_1 = \mathrm{GL}_2(\mathbf{R})$, the group of invertible 2×2 matrices with real entries with the group operation matrix multiplication. We know that $\mathrm{GL}_2(\mathbf{R})$ is not abelian.

Let $G_2 = \mathbf{R}^\times$, the set of non-zero real numbers, with group operation multiplication, and we know that $\mathbf{R}^\times$ is abelian.

Let $\varphi(A) = \det(A)$. We know that the determinant function is a homomorphism, and the equation $\det \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} = x$ shows that any $x \in \mathbf{R}^\times$ is the determinant of some matrix, proving that the homomorphism is surjective.

(b) This statement is also *false*. One example is the homomorphism from $\mathbf{Z}/3\mathbf{Z}$ to S_3 that came from a recent homework assignment. Let $\varphi(0) = e$, $\varphi(1) = (123)$, and $\varphi(2) = (132)$. This is an injective homomorphism and $\mathbf{Z}/3\mathbf{Z}$ is abelian, but S_3 is not abelian.

3. (*15 points*) Find a non-trivial group homomorphism from S_3 to $\mathbf{Z}/6\mathbf{Z}$, or show that no such homomorphism exists.

Answer: The homomorphism can't be an isomorphism, because S_3 is not abelian and $\mathbf{Z}/6\mathbf{Z}$ is abelian. That means that there must be a kernel, and the kernel must be a normal subgroup. The homomorphism that results is $\varphi(e) = \varphi(123) = \varphi(132) = 0$ and $\varphi(12) = \varphi(13) = \varphi(23) = 3$.

4. (*15 points*) Find a non-trivial group homomorphism from $\mathbf{Z}/6\mathbf{Z}$ to S_3, or show that no such homomorphism exists.

Answer: There are five possibilities here, though again the homomorphism cannot be a bijection. The possibilities are determined by $\varphi(1)$, and they are:

$$\varphi_1 : \begin{array}{l} 0 \to e \\ 1 \to (12) \\ 2 \to e \\ 3 \to (12) \\ 4 \to e \\ 5 \to (12) \end{array} \quad \varphi_2 : \begin{array}{l} 0 \to e \\ 1 \to (13) \\ 2 \to e \\ 3 \to (13) \\ 4 \to e \\ 5 \to (13) \end{array} \quad \varphi_3 : \begin{array}{l} 0 \to e \\ 1 \to (23) \\ 2 \to e \\ 3 \to (23) \\ 4 \to e \\ 5 \to (23) \end{array} \quad \varphi_4 : \begin{array}{l} 0 \to e \\ 1 \to (123) \\ 2 \to (132) \\ 3 \to e \\ 4 \to (123) \\ 5 \to (132) \end{array} \quad \varphi_5 : \begin{array}{l} 0 \to e \\ 1 \to (132) \\ 2 \to (123) \\ 3 \to e \\ 4 \to (132) \\ 5 \to (123) \end{array}$$

5. (*15 points*) Find a non-trivial ring homomorphism from $\mathbf{Z}/2\mathbf{Z}$ to $\mathbf{Z}/4\mathbf{Z}$, or show that no such homomorphism exists.

Answer: We must have $\varphi(1) = 1$ in order to satisfy our definition of a ring homomorphism. But then $\varphi(1+1) = 1+1 = 2$, while $\varphi(1+1) = \varphi(0) = 0$. This is a contradiction, so there is no ring homomorphism.

6. (*15 points*) Find a non-trivial ring homomorphism from $\mathbf{Z}/8\mathbf{Z}$ to $\mathbf{Z}/4\mathbf{Z}$, or show that no such homomorphism exists.

Answer: Again, we must have $\varphi(1) = 1$, forcing $\varphi(2) = 2$, $\varphi(3) = 3$, $\varphi(4) = 0$, $\varphi(5) = 1$, $\varphi(6) = 2$, and $\varphi(7) = 3$. This is a homomorphism, and the kernel is $\{0, 4\}$, which is an ideal of $\mathbf{Z}/8\mathbf{Z}$.

Grade	Number of people
83	1
76	1
68	2
65	1
47	1
40	1
38	1

Mean: 60.62
Standard deviation: 15.76