Mathematics 310
Examination 3
December 7, 2011

Please do all of your work in the blue booklets. Please work clearly and neatly, and label your answers. You do not need to do the problems in order. No credit will be given for answers without explanations.

Cheating will result in a failing grade.

No calculators may be used during this examination.

The problems are not arranged in order of increasing difficulty, so you might want to read all of them before beginning.

All rings contain an identity element.

All ring homomorphisms $\varphi : R_1 \to R_2$ must satisfy $\varphi(1_{R_1}) = 1_{R_2}$.

1. (*20 points*) Let $\mathbf{F}_2 = \mathbf{Z}/2\mathbf{Z}$.
   (a) Show that the polynomial $p(x) = x^2 + x + 1$ is irreducible in $\mathbf{F}_2[x]$.
   (b) How many elements are in the field $E = \mathbf{F}_2[x]/(p(x))$?
   (c) Write out the addition and multiplication tables for E.

2. (*20 points*) Suppose that $f(x), g(x) \in \mathbf{Z}[x]$, with $\deg(f) = \deg(g) = 10$. Suppose as well that $f(0) = g(0)$, $f(1) = g(1)$, $\ldots$, $f(10) = g(10)$. Prove that f and g are the same polynomial.

3. (*20 points*) Let F be a field. Find all ideals of the ring $F \oplus F$ other than $F \oplus F$ and $\{(0, 0)\}$.

4. (*20 points*) Suppose that R_1 and R_2 are groups, and $\varphi : R_1 \to R_2$ is a ring homomorphism. Prove or give a counterexample to each of these two statements:
   (a) If R_2 is an integral domain and φ is surjective, then R_1 is an integral domain.
   (b) If R_2 is an integral domain and φ is injective, then R_1 is an integral domain.

A counterexample requires you to find specific rings R_1 and R_2, as well as the homomorphism φ.

5. (*20 points*) Suppose that $f(x), g(x) \in \mathbf{Q}[x]$ are polynomials, and for some complex number α, $f(\alpha) = g(\alpha) = 0$. Prove that f and g are *not* relatively prime polynomials.