Mathematics 310
Examination 3
December 7, 2011
Answers

1. (*20 points*) Let $\mathbf{F}_2 = \mathbf{Z}/2\mathbf{Z}$.
 (a) Show that the polynomial $p(x) = x^2 + x + 1$ is irreducible in $\mathbf{F}_2[x]$.
 (b) How many elements are in the field $E = \mathbf{F}_2[x]/(p(x))$?
 (c) Write out the addition and multiplication tables for E.

Answer: (a) We compute $p(0) = 1$ and $p(1) = 1$. Therefore, $p(x)$ has no linear factor, so it must be irreducible.

(b) We have $[E : \mathbf{F}_2] = \deg(p) = 2$, and so the number of elements in $E = 2^2 = 4$.

(c) Let $E = \{0, 1, x, x + 1\}$. The addition and multiplication tables are:

$+$	0	1	x	$x+1$
0	0	1	x	$x+1$
1	1	0	$x+1$	x
x	x	$x+1$	0	1
$x+1$	$x+1$	x	1	0

$\times$	0	1	x	$x+1$
0	0	0	0	0
1	0	1	x	$x+1$
x	0	x	$x+1$	1
$x+1$	0	$x+1$	1	x

2. (*20 points*) Suppose that $f(x), g(x) \in \mathbf{Z}[x]$, with $\deg(f) = \deg(g) = 10$. Suppose as well that $f(0) = g(0)$, $f(1) = g(1)$, $\ldots$, $f(10) = g(10)$. Prove that f and g are the same polynomial.

Answer: Let $h(x) = f(x) - g(x)$. We know that either $h = 0$ or $\deg(h) \leq 10$, and we also know that $h(0) = h(1) = h(2) = \cdots = h(10) = 0$. A polynomial of degree 10 cannot have 11 roots, so we must conclude that h is the 0 polynomial, which says that f and g are the same polynomial.

3. (*20 points*) Let F be a field. Find all ideals of the ring $F \oplus F$ other than $F \oplus F$ and $\{(0, 0)\}$.

Answer: We know from a homework problem that $\{(f, 0) : f \in F\}$ and $\{(0, f) : f \in F\}$ are ideals. We now show that those are the only ideals. Suppose that I is an ideal which contains an element (a, b), with $a \neq 0$ and $b \neq 0$. Because $a \neq 0$, we know that a^{-1} exists, so multiplication of (a, b) by $(a^{-1}, 0)$ shows that $(1, 0) \in I$. Similarly, because $b \neq 0$, we can multiply by $(0, b^{-1})$ to see that $(0, 1) \in I$. But if $(1, 0) \in I$, then $(f_1, 0) \in I$ for every $f_1 \in F$. Similarly, because $(0, 1) \in I$, we see that $(0, f_2) \in I$ for every $f_2 \in F$. Because I is closed under addition, we now conclude that $(f_1, f_2) \in I$, so that $I = F \oplus F$.

4. (*20 points*) Suppose that R_1 and R_2 are groups, and $\varphi : R_1 \to R_2$ is a ring homomorphism. Prove or give a counterexample to each of these two statements:
 (a) If R_2 is an integral domain and φ is surjective, then R_1 is an integral domain.
 (b) If R_2 is an integral domain and φ is injective, then R_1 is an integral domain.

A counterexample requires you to find specific rings R_1 and R_2, as well as the homomorphism φ.

Answer: (a) This statement is *false*. One example is the homomorphism $\varphi : \mathbf{Z}/4\mathbf{Z} \to \mathbf{Z}/2\mathbf{Z}$ given by $\varphi(1) = \varphi(3) = 1$ and $\varphi(0) = \varphi(2) = 0$. This is surjective, $\mathbf{Z}/2\mathbf{Z}$ is an integral domain, but $\mathbf{Z}/4\mathbf{Z}$ is not an integral domain.

(b) This statement is *true*. Suppose that $a, b \in R_1$, and $ab = 0$. We need to prove that $a = 0$ or $b = 0$. We know that $\varphi(ab) = 0$, so $\varphi(a)\varphi(b) = 0$. Because R_2 is an integral domain, we know that $\varphi(a) = 0$ or $\varphi(b) = 0$. Because φ is an injection, we now can assert that either $a = 0$ or $b = 0$.

5. (*20 points*) Suppose that $f(x), g(x) \in \mathbf{Q}[x]$ are polynomials, and for some complex number α, $f(\alpha) = g(\alpha) = 0$. Prove that f and g are *not* relatively prime polynomials.

Answer: Suppose that f and g are relatively prime. We can then find polynomials $a(x), b(x) \in \mathbf{Q}[x]$ so that $a(x)f(x) + b(x)g(x) = 1$. Now substitute $x = \alpha$, and we get $a(\alpha) \cdot 0 + b(\alpha) \cdot 0 = 1$. This is a contradiction.

Grade	Number of people
70	1
65	2
50	2
30	1
25	1
15	1

Mean: 46.25
Standard deviation: 19.32