Name______________________________

1	2	3	4	5	6	7	8	9	10	Total

MT310.01: Introduction to Abstract Algebra
Final Examination
Tuesday, December 20, 2011, 9 AM
Gasson 306

Please label your answers clearly, as I will not have time to perform extensive searches for answers. No credit will be given for answers without explanations.

Cheating will result in a failing grade.

Calculators may not be used during this examination.

The problems are not arranged in order of increasing difficulty, so you might want to read all of them before beginning.

1. *(10 points)* The following sets are all commutative rings: $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{C}$, $\mathbf{Z}/2\mathbf{Z}$, $\mathbf{Z}/3\mathbf{Z}$, $\mathbf{Z}/4\mathbf{Z}$, $\mathbf{Z}/5\mathbf{Z}$, and $\mathbf{Z}/6\mathbf{Z}$. Which of these rings are integral domains? Which are fields?

You do not need to justify your answers to this question.

2. *(10 points)* (a) State Eisenstein's Criterion.

(b) Let n be a positive integer, and let k be an integer which is bigger than 1. Prove that $\sqrt[k]{25n+15}$ is irrational.

3. *(10 points)* Suppose that F is a field, and $f(x)$ is an element of $F[x]$. Prove or give a counterexample to each of the following statements:

(a) If $f(x)$ is irreducible, then $f(x^2)$ is irreducible.

(b) If $f(x^2)$ is irreducible, then $f(x)$ is irreducible.

A counterexample means finding a specific field F and specific polynomial $f(x)$ that makes the statement false.

4. *(10 points)* Suppose that $\phi : G_1 \to G_2$ is a homomorphism of groups. Prove or give a counterexample:

(a) If $H_1 \lhd G_1$, then $\phi(H_1) \lhd G_2$

(b) If $H_2 \lhd G_2$, then $\phi^{-1}(H_2) \lhd G_1$.

As usual, $\phi(H_1)$ is defined with the equation

$$\phi(H_1) = \{\phi(h_1) : h_1 \in H_1\}$$

and $\phi^{-1}(H_2)$ is defined with the equation

$$\phi^{-1}(H_2) = \{g_1 \in G_1 : \phi(g_1) \in H_2\}.$$

Finding a counterexample means finding specific groups G_1 and G_2, a specific homomorphism ϕ, and a specific normal subgroup which makes the statement false.

Be sure not to assume in your proofs that ϕ is either injective or surjective.

5. *(10 points)* Suppose that $\phi : R_1 \to R_2$ is a homomorphism of rings. Prove or give a counterexample:

(a) If I_1 is an ideal of R_1, then $\phi(I_1)$ is an ideal in R_2

(b) If I_2 is an ideal of R_2, then $\phi^{-1}(I_2)$ is an ideal in R_1.

Finding a counterexample means finding specific rings R_1 and R_2, a specific homomorphism ϕ, and a specific ideal which makes the statement false.

Be sure not to assume in your proofs that ϕ is either injective or surjective.

6. *(5 points)* Suppose that R is a commutative ring, and the only ideals of R are $\{0\}$ and R. Prove that R is a field.

7. *(10 points)* Let

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 1 & 5 & 2 & 3 & 7 & 6 & 9 & 8 \end{pmatrix} \qquad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 6 & 8 & 9 & 5 & 7 & 2 & 1 & 3 & 4 \end{pmatrix}$$

(a) Write $\sigma \circ \tau$ in cycle notation.
(b) What is the order of σ?
(c) What is the order of τ?
(d) Is σ even or odd?
(e) Is τ even or odd?

Be sure to justify your answers.

8. *(15 points)* Remember that A_4 consists of the 12 elements in S_4 which are even permutations. Let $H = \{e, (12)(34), (13)(24), (14)(23)\}$. You may assume without proof that H is a subgroup of A_4.

(a) Show that H is group isomorphic to $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$.

(b) List the left and right cosets of H in A_4.

(c) Is H a normal subgroup of A_4?

9. *(10 points)* Suppose that G is an abelian group with 6 elements. Show that G is isomorphic to $\mathbf{Z}/6\mathbf{Z}$.

10. *(10 points)* Suppose that G is a nonabelian group with 6 elements. Show that G is isomorphic to S_3.

Hint: Start by explaining why there are elements $a, b \in G$ with $o(a) = 2$ and $o(b) = 3$. Then explain why $G = \{e, b, b^2, a, ab, ab^2\}$. Now, compute ba. Finally, write out a multiplication table for G and compare with the multiplication table for S_3.