Please label your answers clearly, as I will not have time to perform extensive searches for answers. No credit will be given for answers without explanations.

Cheating will result in a failing grade.

Calculators may not be used during this examination.

The problems are not arranged in order of increasing difficulty, so you might want to read all of them before beginning.

1. (10 points) The following sets are all commutative rings: \textbf{Z}, \textbf{Q}, \textbf{R}, \textbf{C}, \textbf{Z}/2\textbf{Z}, \textbf{Z}/3\textbf{Z}, \textbf{Z}/4\textbf{Z}, \textbf{Z}/5\textbf{Z}, and \textbf{Z}/6\textbf{Z}. Which of these rings are integral domains? Which are fields?

You do not need to justify your answers to this question.
2. (10 points) (a) State Eisenstein’s Criterion.

(b) Let $n$ be a positive integer, and let $k$ be an integer which is bigger than 1. Prove that $\sqrt{25n + 15}$ is irrational.
3. (10 points) Suppose that $F$ is a field, and $f(x)$ is an element of $F[x]$. Prove or give a counterexample to each of the following statements:

(a) If $f(x)$ is irreducible, then $f(x^2)$ is irreducible.
(b) If $f(x^2)$ is irreducible, then $f(x)$ is irreducible.

A counterexample means finding a specific field $F$ and specific polynomial $f(x)$ that makes the statement false.
4. (10 points) Suppose that \( \phi : G_1 \to G_2 \) is a homomorphism of groups. Prove or give a counterexample:

(a) If \( H_1 \trianglelefteq G_1 \), then \( \phi(H_1) \trianglelefteq G_2 \).

(b) If \( H_2 \trianglelefteq G_2 \), then \( \phi^{-1}(H_2) \trianglelefteq G_1 \).

As usual, \( \phi(H_1) \) is defined with the equation

\[
\phi(H_1) = \{ \phi(h_1) : h_1 \in H_1 \}
\]

and \( \phi^{-1}(H_2) \) is defined with the equation

\[
\phi^{-1}(H_2) = \{ g_1 \in G_1 : \phi(g_1) \in H_2 \}.
\]

Finding a counterexample means finding specific groups \( G_1 \) and \( G_2 \), a specific homomorphism \( \phi \), and a specific normal subgroup which makes the statement false.

Be sure not to assume in your proofs that \( \phi \) is either injective or surjective.
5. (10 points) Suppose that $\phi : R_1 \rightarrow R_2$ is a homomorphism of rings. Prove or give a counterexample:

(a) If $I_1$ is an ideal of $R_1$, then $\phi(I_1)$ is an ideal in $R_2$
(b) If $I_2$ is an ideal of $R_2$, then $\phi^{-1}(I_2)$ is an ideal in $R_1$.

Finding a counterexample means finding specific rings $R_1$ and $R_2$, a specific homomorphism $\phi$, and a specific ideal which makes the statement false.

Be sure not to assume in your proofs that $\phi$ is either injective or surjective.
6. (5 points) Suppose that $R$ is a commutative ring, and the only ideals of $R$ are $\{0\}$ and $R$. Prove that $R$ is a field.
7. (10 points) Let
\[ \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 1 & 5 & 2 & 3 & 7 & 6 & 9 & 8 \end{pmatrix} \quad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 6 & 8 & 9 & 5 & 7 & 2 & 1 & 3 & 4 \end{pmatrix} \]

(a) Write \( \sigma \circ \tau \) in cycle notation.
(b) What is the order of \( \sigma \)?
(c) What is the order of \( \tau \)?
(d) Is \( \sigma \) even or odd?
(e) Is \( \tau \) even or odd?

Be sure to justify your answers.
8. (15 points) Remember that $A_4$ consists of the 12 elements in $S_4$ which are even permutations. Let $H = \{e, (12)(34), (13)(24), (14)(23)\}$. You may assume without proof that $H$ is a subgroup of $A_4$.

(a) Show that $H$ is group isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

(b) List the left and right cosets of $H$ in $A_4$.

(c) Is $H$ a normal subgroup of $A_4$?
9. (10 points) Suppose that $G$ is an abelian group with 6 elements. Show that $G$ is isomorphic to $\mathbb{Z}/6\mathbb{Z}$. 
10. (10 points) Suppose that $G$ is a nonabelian group with 6 elements. Show that $G$ is isomorphic to $S_3$.

*Hint:* Start by explaining why there are elements $a, b \in G$ with $o(a) = 2$ and $o(b) = 3$. Then explain why $G = \{e, b, b^2, a, ab, ab^2\}$. Now, compute $ba$. Finally, write out a multiplication table for $G$ and compare with the multiplication table for $S_3$. 