Mathematics 310
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Homework 1
Due September 16, 2011

Remember that the Fibonacci numbers are defined with the three equations

$$F_1 = 1$$
$$F_2 = 1$$
$$F_n = F_{n-1} + F_{n-2}$$

For example, we have $F_3 = 2$, $F_4 = 3$, and $F_5 = 5$.

1. Let $k$ be a positive integer. Prove that $F_{3k}$ is always even.

2. Let $k$ be a positive integer. Prove that $F_{4k}$ is always a multiple of 3.

3. Suppose that $G$ is a group, and for every element $a \in G$, we have $a = a^{-1}$. Prove that $G$ must be abelian.

4. If $G$ is a finite group of *even* order, show that there must be an element $a \neq e$ such that $a = a^{-1}$.

5. Suppose that $G$ is a group in which $(ab)^2 = a^2b^2$ for every pair of elements $a$ and $b$ in $G$. Prove that $G$ must be abelian.

6. If $A$ and $B$ are subgroups of $G$, show that $A \cap B$ is a subgroup of $G$.

7. Let $G$ be a group in which $(ab)^3 = a^3b^3$ and $(ab)^5 = a^5b^5$ for all $a, b \in G$. Show that $G$ is abelian.

8. Suppose that $G$ is a group in which for some fixed positive integer $n$, we have the three equations

$$(ab)^n = a^nb^n$$
$$(ab)^{n+1} = a^{n+1}b^{n+1}$$
$$(ab)^{n+2} = a^{n+2}b^{n+2}$$

for every pair of elements $a$ and $b$ in $G$. Prove that $G$ must be abelian.

9. Verify that $Z(G)$, the center of $G$, is a subgroup of $G$.

10. If $G$ is an abelian group and if $H = \{a \in G \mid a^2 = e\}$, show that $H$ is a subgroup of $G$.