Mathematics 310
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Homework 1
Answers

Remember that the Fibonacci numbers are defined with the three equations

$$F_1 = 1$$
$$F_2 = 1$$
$$F_n = F_{n-1} + F_{n-2}$$

For example, we have $F_3 = 2$, $F_4 = 3$, and $F_5 = 5$.

1. Let k be a positive integer. Prove that F_{3k} is always even.

Answer: We prove this using induction. When $k = 1$, we must show that F_3 is even, which is true because $F_3 = 2$. Now, we assume that F_{3k} is even, and we must prove that F_{3k+3} is even. We have

$$F_{3k+3} = F_{3k+2} + F_{3k+1} = (F_{3k+1} + F_{3k}) + F_{3k+1} = 2F_{3k+1} + F_{3k}$$

Because $2F_{3k+1}$ is even, and the inductive hypothesis is that F_{3k} is even, we can conclude that F_{3k+3} must be even.

2. Let k be a positive integer. Prove that F_{4k} is always a multiple of 3.

Answer: Again, we proceed by induction. When $k = 1$, we must show that F_4 is a multiple of 3, which is true because $F_4 = 3$. Now, we assume that F_{4k} is a multiple of 3, and we must prove that F_{4k+4} is a multiple of 3. We have

$$\begin{aligned} F_{4k+4} &= F_{4k+3} + F_{4k+2} = (F_{4k+2} + F_{4k+1}) + F_{4k+2} \\ &= 2F_{4k+2} + F_{4k+1} = 2(F_{4k+1} + F_{4k}) + F_{4k+1} = 3F_{4k+1} + 2F_{4k}. \end{aligned}$$

We know that $3F_{4k+1}$ is a multiple of 3, and by assumption F_{4k} is a multiple of 3, and therefore F_{4k+4} is also a multiple of 3.

3. Suppose that G is a group, and for every element $a \in G$, we have $a = a^{-1}$. Prove that G must be abelian.

Answer: Let $a, b \in G$. We know that $(ab)^{-1} = b^{-1}a^{-1}$, and the given information tells us both that $(ab)^{-1} = ab$ and that $b^{-1}a^{-1} = ba$. Therefore, $ab = ba$, and the group G is abelian.

4. If G is a finite group of *even* order, show that there must be an element $a \neq e$ such that $a = a^{-1}$.

Answer: We can match each element in G with its inverse. Because $g = (g^{-1})^{-1}$, each element is paired with at most one element. However, the identity element e is paired with itself, because $e = e^{-1}$. Because there are an even number of elements in the set G, there must also be at least one other element which is paired with itself, which is just another way of saying that there is another element $g \in G$ with $g = g^{-1}$.

5. Suppose that G is a group in which $(ab)^2 = a^2b^2$ for every pair of elements a and b in G. Prove that G must be abelian.

Answer: Let $a, b \in G$. Then we are given $(ab)^2 = a^2b^2$, but on the other hand, the definition of $(ab)^2$ tells us that $(ab)^2 = abab$. Therefore, we have $abab = a^2b^2$. Cancel a factor of a on the left and a factor of b on the right, and we have $ba = ab$, which shows that G is abelian.

6. If A and B are subgroups of G, show that $A \cap B$ is a subgroup of G.

Answer: We know that $e \in A$ and $e \in B$, so $e \in A \cap B$.

Second, suppose that $g, h \in A \cap B$. Then $g, h \in A$, and because A is a subgroup, we know that $gh \in A$. Similarly, $gh \in B$. Therefore, $gh \in A \cap B$, which shows that $A \cap B$ is closed under the group operation.

Third, suppose that $g \in A \cap B$. We know that $g^{-1} \in A$ and $g^{-1} \in B$, and therefore $g^{-1} \in A \cap B$, and therefore $A \cap B$ contains inverses of all of its elements.

This shows that $A \cap B$ is a subgroup of G.

7. Let G be a group in which $(ab)^3 = a^3b^3$ and $(ab)^5 = a^5b^5$ for all $a, b \in G$. Show that G is abelian.

Answer: The equation $(ab)^3 = a^3b^3$ tells us that $ababab = aaabbb$. Cancel a factor of a on the left and b on the right, and we have $baba = aabb$. Similarly, $(ab)^5 = a^5b^5$ tells us that $ababababab = aaaaabbbbb$, and cancellation yields $babababa = aaaabbbb$. Group this as $(baba)(baba) = aaaabbbb$, and use the equation $baba = aabb$ to get $aabbaabb = aaaabbbb$. Now, we can cancel two factors of a on the left and two factors of b on the right to get $bbaa = aabb$. But $aabb = baba$, so we have $bbaa = baba$, and now cancellation of b on the left and a on the right yields $ba = ab$, which shows that G is abelian.

8. Suppose that G is a group in which for some fixed positive integer n, we have the three equations

$$\begin{aligned}(ab)^n &= a^nb^n\\ (ab)^{n+1} &= a^{n+1}b^{n+1}\\ (ab)^{n+2} &= a^{n+2}b^{n+2}\end{aligned}$$

for every pair of elements a and b in G. Prove that G must be abelian.

Answer: Take the first equation, and multiply by ab on the right. We get $(ab)^{n+1} = a^nb^nab$. Substitute into the equation $(ab)^{n+1} = a^{n+1}b^{n+1}$ to get $a^nb^nab = a^{n+1}b^{n+1}$. Cancel a factor of a^n on the left, and b on the right, and we get $b^na = ab^n$.

Return to the first given equation, and multiply by $abab$ on the right. We get $(ab)^{n+2} = a^nb^nabab$. Substitute into the third given equation, and the result is $a^nb^nabab = a^{n+2}b^{n+2}$. Now cancellation results in $(b^na)ba = a^2b^{n+1}$. Because $b^na = ab^n$, we can substitute and get $(ab^n)ba = a^2b^{n+1}$, and now cancellation of a factor of a on the left yields $b^{n+1}a = ab^{n+1}$. Rewrite this as $b(b^na) = ab^{n+1}$. Now substitute $b^na = ab^n$, and we get $bab^n = ab^{n+1}$.

Finally, cancel a factor of b^n on the right, and the result is $ba = ab$, which shows that G is abelian.

9. Verify that $Z(G)$, the center of G, is a subgroup of G.

Answer: Recall that the definition is

$$Z(G) = \{g \in G : gx = xg \text{ for all } x \in G\}.$$

First, because $ex = xe$ for all $x \in G$, we have $e \in Z(G)$.

Second, suppose that $g, h \in Z(G)$, so that $gx = xg$ and $hx = xh$ for all $x = inG$. Then $(gh)x = g(hx) = g(xh) = (gx)h = (xg)h = x(gh)$, which shows that $gh \in Z(G)$.

Third, if $g \in Z(G)$, then $gx = xg$ for every $x \in G$. Multiply this equation on both the left and the right by g^{-1}, and the resulting equation is $xg^{-1} = g^{-1}x$ for every $x \in G$. This shows that $x^{-1} \in Z(G)$, showing that the inverse of every element in $Z(G)$ is also in $Z(G)$.

These three properties show that $Z(G)$ is a subgroup.

10. If G is an abelian group and if $H = \{a \in G \mid a^2 = e\}$, show that H is a subgroup of G.

Answer: First, $e^2 = e$, so $e \in H$.

Second, if $g, h \in H$, then $g^2 = e$ and $h^2 = e$. Using the fact that $gh = hg$, we have $(gh)^2 = g^2h^2 = e$, showing that H is closed under the group operation.

Finally, if $h \in H$, then $(h^{-1})^2 = (h^2)^{-1} = e$, which shows that the inverse of each element in H is also in H. That shows that H is a subgroup.