Remember that the Fibonacci numbers are defined with the three equations
\[ F_1 = 1 \]
\[ F_2 = 1 \]
\[ F_n = F_{n-1} + F_{n-2} \]
For example, we have \( F_3 = 2 \), \( F_4 = 3 \), and \( F_5 = 5 \).

1. Let \( k \) be a positive integer. Prove that \( F_{3k} \) is always even.

**Answer:** We prove this using induction. When \( k = 1 \), we must show that \( F_3 \) is even, which is true because \( F_3 = 2 \). Now, we assume that \( F_{3k} \) is even, and we must prove that \( F_{3k+3} \) is even. We have
\[ F_{3k+3} = F_{3k+2} + F_{3k+1} = (F_{3k+1} + F_{3k}) + F_{3k+1} = 2F_{3k+1} + F_{3k} \]
Because \( 2F_{3k+1} \) is even, and the inductive hypothesis is that \( F_{3k} \) is even, we can conclude that \( F_{3k+3} \) must be even.

2. Let \( k \) be a positive integer. Prove that \( F_{4k} \) is always a multiple of 3.

**Answer:** Again, we proceed by induction. When \( k = 1 \), we must show that \( F_4 \) is a multiple of 3, which is true because \( F_4 = 3 \). Now, we assume that \( F_{4k} \) is a multiple of 3, and we must prove that \( F_{4k+4} \) is a multiple of 3. We have
\[ F_{4k+4} = F_{4k+3} + F_{4k+2} = (F_{4k+2} + F_{4k+1}) + F_{4k+2} \\
= 2F_{4k+2} + F_{4k+1} = 2(F_{4k+1} + F_{4k}) + F_{4k+1} = 3F_{4k+1} + 2F_{4k} \]
We know that \( 3F_{4k+1} \) is a multiple of 3, and by assumption \( F_{4k} \) is a multiple of 3, and therefore \( F_{4k+4} \) is also a multiple of 3.

3. Suppose that \( G \) is a group, and for every element \( a \in G \), we have \( a = a^{-1} \). Prove that \( G \) must be abelian.

**Answer:** Let \( a, b \in G \). We know that \( (ab)^{-1} = b^{-1}a^{-1} \), and the given information tells us both that \( (ab)^{-1} = ab \) and that \( b^{-1}a^{-1} = ba \). Therefore, \( ab = ba \), and the group \( G \) is abelian.

4. If \( G \) is a finite group of even order, show that there must be an element \( a \neq e \) such that \( a = a^{-1} \).

**Answer:** We can match each element in \( G \) with its inverse. Because \( g = (g^{-1})^{-1} \), each element is paired with at most one element. However, the identity element \( e \) is paired with itself, because \( e = e^{-1} \). Because there are an even number of elements in the set \( G \), there must also be at least one other element which is paired with itself, which is just another way of saying that there is another element \( g \in G \) with \( g = g^{-1} \).

5. Suppose that \( G \) is a group in which \( (ab)^2 = a^2b^2 \) for every pair of elements \( a \) and \( b \) in \( G \). Prove that \( G \) must be abelian.
Let \( a, b \in G \). Then we are given \((ab)^2 = a^2b^2\), but on the other hand, the definition of \((ab)^2\) tells us that \((ab)^2 = abab\). Therefore, we have \(abab = a^2b^2\). Cancel a factor of \(a\) on the left and a factor of \(b\) on the right, and we have \(ba = ab\), which shows that \(G\) is abelian.

6. If \(A\) and \(B\) are subgroups of \(G\), show that \(A \cap B\) is a subgroup of \(G\).

Answer: We know that \(e \in A\) and \(e \in B\), so \(e \in A \cap B\).

Second, suppose that \(g, h \in A \cap B\). Then \(g, h \in A\), and because \(A\) is a subgroup, we know that \(gh \in A\). Similarly, \(gh \in B\). Therefore, \(gh \in A \cap B\), which shows that \(A \cap B\) is closed under the group operation.

Third, suppose that \(g \in A \cap B\). We know that \(g^{-1} \in A\) and \(g^{-1} \in B\), and therefore \(g^{-1} \in A \cap B\), and therefore \(A \cap B\) contains inverses of all of its elements.

This shows that \(A \cap B\) is a subgroup of \(G\).

7. Let \(G\) be a group in which \((ab)^3 = a^3b^3\) and \((ab)^5 = a^5b^5\) for all \(a, b \in G\). Show that \(G\) is abelian.

Answer: The equation \((ab)^3 = a^3b^3\) tells us that \(ababab = aaabbb\). Cancel a factor of \(a\) on the left and \(b\) on the right, and we have \(baba = aabb\). Similarly, \((ab)^5 = a^5b^5\) tells us that \(ababababab = aaaaaabbbb\), and cancellation yields \(bababa = aaaaabbb\). Group this as \((ab)(ab) = aaaaabbb\), and use the equation \(baba = aabb\) to get \(aabbaabb = aaaaabbb\). Now, we can cancel two factors of \(a\) on the left and two factors of \(b\) on the right to get \(bbaa = aabb\).

But \(aabb = bab\), so we have \(bba = baba\), and now cancellation of \(b\) on the left and \(a\) on the right yields \(ba = ab\), which shows that \(G\) is abelian.

8. Suppose that \(G\) is a group in which for some fixed positive integer \(n\), we have the three equations

\[
(ab)^n = a^n b^n \\
(ab)^{n+1} = a^{n+1} b^{n+1} \\
(ab)^{n+2} = a^{n+2} b^{n+2}
\]

for every pair of elements \(a\) and \(b\) in \(G\). Prove that \(G\) must be abelian.

Answer: Take the first equation, and multiply by \(ab\) on the right. We get \((ab)^{n+1} = a^n b^n ab\). Substitute into the equation \((ab)^{n+1} = a^{n+1} b^{n+1}\) to get \(a^n b^n ab = a^{n+1} b^{n+1}\). Cancel a factor of \(a^n\) on the left, and \(b\) on the right, and we get \(b^n a = ab^n\).

Return to the first given equation, and multiply by \(abab\) on the right. We get \((ab)^{n+2} = a^n b^n abab\). Substitute into the third given equation, and the result is \(a^n b^n abab = a^{n+2} b^{n+2}\). Now cancellation results in \((b^n) a b a = a^2 b^{n+1}\). Because \(b^n a = ab^n\), we can substitute and get \((ab^n) a b = a^2 b^{n+1}\), and now cancellation of a factor of \(a\) on the left yields \(b^{n+1} a = ab^{n+1}\).

Rewrite this as \(b(b^n a) = ab^{n+1}\). Now substitute \(b^n a = ab^n\), and we get \(bab^n = ab^{n+1}\).

Finally, cancel a factor of \(b^n\) on the right, and the result is \(ba = ab\), which shows that \(G\) is abelian.

9. Verify that \(Z(G)\), the center of \(G\), is a subgroup of \(G\).

Answer: Recall that the definition is

\[
Z(G) = \{ g \in G : gx = xg \text{ for all } x \in G \}.
\]

First, because \(ex = xe\) for all \(x \in G\), we have \(e \in Z(G)\).
Second, suppose that \( g, h \in Z(G) \), so that \( gx = xg \) and \( hx = xh \) for all \( x \in \text{in} G \). Then 
\[(gh)x = g(hx) = g(xh) = (gx)h = (xg)h = x(gh),\]
which shows that \( gh \in Z(G) \).

Third, if \( g \in Z(G) \), then \( gx = xg \) for every \( x \in G \). Multiply this equation on both the left and the right by \( g^{-1} \), and the resulting equation is \( xg^{-1} = g^{-1}x \) for every \( x \in G \). This shows that \( x^{-1} \in Z(G) \), showing that the inverse of every element in \( Z(G) \) is also in \( Z(G) \).

These three properties show that \( Z(G) \) is a subgroup.

10. If \( G \) is an abelian group and if \( H = \{ a \in G \mid a^2 = e \} \), show that \( H \) is a subgroup of \( G \).

Answer: First, \( e^2 = e \), so \( e \in H \).
Second, if \( g, h \in H \), then \( g^2 = e \) and \( h^2 = e \). Using the fact that \( gh = hg \), we have
\[(gh)^2 = g^2h^2 = e,\]
showing that \( H \) is closed under the group operation.
Finally, if \( h \in H \), then \((h^{-1})^2 = (h^2)^{-1} = e\), which shows that the inverse of each element in \( H \) is also in \( H \). That shows that \( H \) is a subgroup.