Mathematics 310
Robert Gross
Homework 2
Due September 23, 2011

1. Let G be a group and H a subgroup of G. Define, for $a, b \in G$, $a \sim b$ if $a^{-1}b \in H$. Prove that this defines an equivalence relation on G, and show that $[a] = aH = \{ah \mid h \in H\}$. The sets aH are called *left cosets* of H in G.

2. Remember that S_3 is another name for the set of all bijections from the set $\{1, 2, 3\}$ to itself. For the sake of the next few problems, let's label the 6 bijections as follows:

$$e: \begin{matrix} 1 \to 1 \\ 2 \to 2 \\ 3 \to 3 \end{matrix} \qquad f: \begin{matrix} 1 \to 2 \\ 2 \to 3 \\ 3 \to 1 \end{matrix} \qquad f^2: \begin{matrix} 1 \to 3 \\ 2 \to 1 \\ 3 \to 2 \end{matrix}$$

$$g: \begin{matrix} 1 \to 2 \\ 2 \to 1 \\ 3 \to 3 \end{matrix} \qquad h: \begin{matrix} 1 \to 3 \\ 2 \to 2 \\ 3 \to 1 \end{matrix} \qquad k: \begin{matrix} 1 \to 1 \\ 2 \to 3 \\ 3 \to 2 \end{matrix}$$

Let H be the subgroup $\{e, g\}$. (You do not need to show that H is a subgroup.) List the elements in each of the 3 right cosets Ha.

3. List the elements in the 3 left cosets aH.

4. On last week's homework, we showed that if G is an abelian group and $H = \{g \in G \mid g^2 = e\}$, then H is a subgroup of G. This fact is only true of abelian groups. Verify that $H = \{a \in S_3 \mid a^2 = e\}$ is *not* a subgroup of S_3.

5. If A and B are subgroups of an abelian group G, let $AB = \{ab \mid a \in A, b \in B\}$. Prove that AB is a subgroup of G.

6. Now find an example of a group G and two subgroups A and B of G such that AB is *not* a subgroup of G.

7. If in a group G, $aba^{-1} = b^i$, show that $a^r b a^{-r} = b^{i^r}$ for all positive integers r.

8. Suppose that G is a group, $a, b \in G$, and

$$a^5 = e \tag{1}$$

$$aba^{-1} = b^2 \tag{2}$$

$$b \neq e \tag{3}$$

What is $o(b)$?

9. Let

$$G = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \,\middle|\, a, b \in \mathbf{R},\ a^2 + b^2 \neq 0 \right\}.$$

Show that G is an abelian group with group operation matrix multiplication.