Mathematics 310
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Homework 2
Answers

1. Let G be a group and H a subgroup of G. Define, for $a, b \in G$, $a \sim b$ if $a^{-1}b \in H$. Prove that this defines an equivalence relation on G, and show that $[a] = aH = \{ah \mid h \in H\}$. The sets aH are called *left cosets* of H in G.

Answer: Reflexivity: We need to check that $a \sim a$. Because $a^{-1}a \in H$, we know that $a \sim a$.

Symmetry: Given $a \sim b$, we need to verify that $b \sim a$. We are given $a^{-1}b \in H$. Because H is a subgroup, it contains the inverse of each of its elements, and so $(a^{-1}b)^{-1} \in H$. But $(a^{-1}b)^{-1} = b^{-1}a$, and if $b^{-1}a \in H$, then $b \sim a$.

Transitivity: Given $a \sim b$ and $b \sim c$, we need to see that $a \sim c$. We are given $a^{-1}b \in H$ and $b^{-1}c \in H$. Because H is closed under the group operation, we know that $(a^{-1}b)(b^{-1}c) = a^{-1}c \in H$, which tells us that $a \sim c$.

Now, if $a \sim b$, then $a^{-1}b = h$ for some $h \in H$, and then $b = ah$. On the other hand, if $b = ah$, then $a^{-1}b \in H$. This shows that $[a] = aH$.

2. Remember that S_3 is another name for the set of all bijections from the set $\{1, 2, 3\}$ to itself. For the sake of the next few problems, let's label the 6 bijections as follows:

$$e: \begin{array}{c} 1 \to 1 \\ 2 \to 2 \\ 3 \to 3 \end{array} \qquad f: \begin{array}{c} 1 \to 2 \\ 2 \to 3 \\ 3 \to 1 \end{array} \qquad f^2: \begin{array}{c} 1 \to 3 \\ 2 \to 1 \\ 3 \to 2 \end{array}$$

$$g: \begin{array}{c} 1 \to 2 \\ 2 \to 1 \\ 3 \to 3 \end{array} \qquad h: \begin{array}{c} 1 \to 3 \\ 2 \to 2 \\ 3 \to 1 \end{array} \qquad k: \begin{array}{c} 1 \to 1 \\ 2 \to 3 \\ 3 \to 2 \end{array}$$

Let H be the subgroup $\{e, g\}$. (You do not need to show that H is a subgroup.) List the elements in each of the 3 right cosets Ha.

Answer: We start by computing that $gf = k$, and that $gf^2 = h$. Therefore, the three right cosets are

$$\begin{aligned} He &= \{e, g\} = Hg \\ Hf &= \{f, k\} = Hk \\ Hf^2 &= \{f^2, h\} = Hh. \end{aligned}$$

3. List the elements in the 3 left cosets aH.

Answer: Now, we compute that $fg = h$, and $f^2g = k$. Therefore, the three left cosets are

$$\begin{aligned} eH &= \{e, g\} = gH \\ fH &= \{f, h\} = hH \\ f^2H &= \{f^2, k\} = kH. \end{aligned}$$

4. On last week's homework, we showed that if G is an abelian group and $H = \{g \in G \mid g^2 = e\}$, then H is a subgroup of G. This fact is only true of abelian groups. Verify that $H = \{a \in S_3 \mid a^2 = e\}$ is *not* a subgroup of S_3.

Answer: We compute that the set $H = \{a \in S_3 \mid a^2 = e\} = \{e, g, h, k\}$. This can't possibly be a subgroup of G, because it has 4 elements and G has 6 elements. Specifically, H is not closed under the group operation, because $gh = f^2 \notin H$.

5. If A and B are subgroups of an abelian group G, let $AB = \{ab \mid a \in A, b \in B\}$. Prove that AB is a subgroup of G.

Answer: First, $e \in A$ and $e \in B$, so $e = e \cdot e \in AB$.

Second, if $a_1b_1, a_2b_2 \in AB$, then $(a_1b_1)(a_2b_2) = (a_1a_2)(b_1b_2) \in AB$, so AB is closed under the group operation.

Third, if $ab \in AB$, then $(ab)^{-1} = a^{-1}b^{-1} \in AB$, so AB contains inverses.

Notice that both closure and inverses used the given information that G be abelian.

6. Now find an example of a group G and two subgroups A and B of G such that AB is *not* a subgroup of G.

Answer: The only non-abelian group that we can easily handle is S_3 (though finding an example with matrices is certainly possible). Let $A = \{e, g\}$ and $B = \{e, h\}$. Then $AB = \{e, g, h, gh = f^2\}$. This set does not contain the inverse of each element, because $(f^2)^{-1} = f$. It also is not closed under the group operation, because $f^2 g = k \notin AB$.

7. If in a group G, $aba^{-1} = b^i$, show that $a^r ba^{-r} = b^{i^r}$ for all positive integers r.

Answer: This problem calls for a proof by induction. The case $r = 1$ is just the given equation.

Second suppose that the statement is true when $r = k$ and we need to prove it when $r = k+1$. We are given that $a^k ba^{-k} = b^{i^k}$. Then $a^{k+1}ba^{-k-1} = a(a^k ba^{-k})a^{-1} = ab^{i^k}a^{-1}$.

Now, $a(cd)a^{-1} = (aca^{-1})(ada^{-1})$, and so $ab^{i^k}a^{-1} = (aba^{-1})^{i^k} = (b^i)^{i^k} = b^{i^{k+1}}$, proving the inductive step.

8. Suppose that G is a group, $a, b \in G$, and

$$a^5 = e \tag{1}$$

$$aba^{-1} = b^2 \tag{2}$$

$$b \neq e \tag{3}$$

What is $o(b)$?

Answer: Using the previous problem, we have $b = ebe = a^5 ba^{-5} = b^{2^5} = b^{32}$. Multiply by b^{-1}, and we have $b^{31} = e$.

We showed in class that if $o(a) = n$ and $a^k = e$, then $n|e$. Here, we have $b^{31} = e$, so $o(b)|31$. Because 31 is prime, we know that $o(b) = 1$ or $o(b) = 31$. Because $b \neq e$, we know that $o(b) \neq 1$, and therefore $o(b) = 31$.

9. Let

$$G = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \middle| \, a, b \in \mathbf{R}, \; a^2 + b^2 \neq 0 \right\}.$$

Show that G is an abelian group with group operation matrix multiplication.

Answer: It's easy to see that $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in G$. Therefore G contains the identity element.j

We need to verify closure. Suppose that $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ and $B = \begin{pmatrix} c & d \\ -d & c \end{pmatrix}$. Then $AB = \begin{pmatrix} ac - bd & ad + bc \\ -ad - bc & -bd + ac \end{pmatrix}$. Note that the resulting matrix again has the form $\begin{pmatrix} t & s \\ -s & t \end{pmatrix}$. Furthermore, the equation $\det(AB) = \det(A)\det(B)$ tells us that $\det(AB) \neq 0$. Hence, $AB \in G$.

Next, $A^{-1} = \frac{1}{a^2 + b^2} \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \in G$.

Finally, to see that G is abelian, we compute $BA = \begin{pmatrix} ca - db & cb + ad \\ -ad - bc & -bc + ac \end{pmatrix} = AB$.