1. Let $G$ be a group and $H$ a subgroup of $G$. Define, for $a, b \in G$, $a \sim b$ if $a^{-1}b \in H$. Prove that this defines an equivalence relation on $G$, and show that $[a] = aH = \{ah \mid h \in H\}$. The sets $aH$ are called left cosets of $H$ in $G$.

**Answer:**

**Reflexivity:** We need to check that $a \sim a$. Because $a^{-1}a \in H$, we know that $a \sim a$.

**Symmetry:** Given $a \sim b$, we need to verify that $b \sim a$. We are given $a^{-1}b \in H$. Because $H$ is a subgroup, it contains the inverse of each of its elements, and so $(a^{-1}b)^{-1} \in H$. But $(a^{-1}b)^{-1} = b^{-1}a$, and if $b^{-1}a \in H$, then $b \sim a$.

**Transitivity:** Given $a \sim b$ and $b \sim c$, we need to see that $a \sim c$. We are given $a^{-1}b \in H$ and $b^{-1}c \in H$. Because $H$ is closed under the group operation, we know that $(a^{-1}b)(b^{-1}c) = a^{-1}c \in H$, which tells us that $a \sim c$.

Now, if $a \sim b$, then $a^{-1}b = h$ for some $h \in H$, and then $b = ah$. On the other hand, if $b = ah$, then $a^{-1}b \in H$. This shows that $[a] = aH$.

2. Remember that $S_3$ is another name for the set of all bijections from the set $\{1, 2, 3\}$ to itself. For the sake of the next few problems, let’s label the 6 bijections as follows:

   \[
   \begin{align*}
   &1 \rightarrow 1 & 1 \rightarrow 2 & 1 \rightarrow 3 \\
   &e : 2 \rightarrow 2 & f : 2 \rightarrow 3 & f^2 : 2 \rightarrow 1 \\
   &3 \rightarrow 3 & 3 \rightarrow 1 & 3 \rightarrow 2
   \end{align*}
   \]

   \[
   \begin{align*}
   &1 \rightarrow 2 & 1 \rightarrow 3 & 1 \rightarrow 1 \\
   &g : 2 \rightarrow 1 & h : 2 \rightarrow 2 & k : 2 \rightarrow 3 \\
   &3 \rightarrow 3 & 3 \rightarrow 1 & 3 \rightarrow 2
   \end{align*}
   \]

Let $H$ be the subgroup $\{e, g\}$. (You do not need to show that $H$ is a subgroup.) List the elements in each of the 3 right cosets $Ha$.

**Answer:** We start by computing that $gf = k$, and that $gf^2 = h$. Therefore, the three right cosets are

   \[
   \begin{align*}
   He &= \{e, g\} = Hg \\
   Hf &= \{f, k\} = Hk \\
   Hf^2 &= \{f^2, h\} = Hh.
   \end{align*}
   \]

3. List the elements in the 3 left cosets $aH$.

**Answer:** Now, we compute that $fg = h$, and $f^2g = k$. Therefore, the three left cosets are

   \[
   \begin{align*}
   eH &= \{e, g\} = gH \\
   fH &= \{f, h\} = hH \\
   f^2H &= \{f^2, k\} = kH.
   \end{align*}
   \]
4. On last week’s homework, we showed that if $G$ is an abelian group and $H = \{ g \in G \mid g^2 = e \}$, then $H$ is a subgroup of $G$. This fact is only true of abelian groups. Verify that $H = \{ a \in S_3 \mid a^2 = e \}$ is not a subgroup of $S_3$.

Answer: We compute that the set $H = \{ a \in S_3 \mid a^2 = e \} = \{ e, g, h, k \}$. This can’t possibly be a subgroup of $G$, because it has 4 elements and $G$ has 6 elements. Specifically, $H$ is not closed under the group operation, because $gh = f^2 \notin H$.

5. If $A$ and $B$ are subgroups of an abelian group $G$, let $AB = \{ ab \mid a \in A, b \in B \}$. Prove that $AB$ is a subgroup of $G$.

Answer: First, $e \in A$ and $e \in B$, so $e = e \cdot e \in AB$.

Second, if $a_1b_1, a_2b_2 \in AB$, then $(a_1b_1)(a_2b_2) = (a_1a_2)(b_1b_2) \in AB$, so $AB$ is closed under the group operation.

Third, if $ab \in AB$, then $(ab)^{-1} = a^{-1}b^{-1} \in AB$, so $AB$ contains inverses.

Notice that both closure and inverses used the given information that $G$ be abelian.

6. Now find an example of a group $G$ and two subgroups $A$ and $B$ of $G$ such that $AB$ is not a subgroup of $G$.

Answer: The only non-abelian group that we can easily handle is $S_3$ (though finding an example with matrices is certainly possible). Let $A = \{ e, g \}$ and $B = \{ e, h \}$. Then $AB = \{ e, g, h, gh = f^2 \}$. This set does not contain the inverse of each element, because $(f^2)^{-1} = f$. It also is not closed under the group operation, because $f^2g = k \notin AB$.

7. If in a group $G$, $aba^{-1} = b^r$, show that $a^rb^{-r} = b^r$ for all positive integers $r$.

Answer: This problem calls for a proof by induction. The case $r = 1$ is just the given equation.

Second suppose that the statement is true when $r = k$ and we need to prove it when $r = k + 1$. We are given that $a^kba^{-k} = b^k$. Then $a^{k+1}ba^{-k-1} = a(a^kba^{-k})a^{-1} = ab^k a^{-1}$.

Now, $a(cd)a^{-1} = (aca^{-1})(ada^{-1})$, and so $ab^k a^{-1} = (aba^{-1})^k = (b^r)^k = b^{kr}$, proving the inductive step.

8. Suppose that $G$ is a group, $a, b \in G$, and

\begin{align*}
(1) & \quad a^5 = e \\
(2) & \quad aba^{-1} = b^2 \\
(3) & \quad b \neq e
\end{align*}

What is $o(b)$?

Answer: Using the previous problem, we have $b = ebe = a^5ba^{-5} = b^{5^5} = b^{32}$. Multiply by $b^{-1}$, and we have $b^{31} = e$.

We showed in class that if $o(a) = n$ and $a^k = e$, then $n|e$. Here, we have $b^{31} = e$, so $o(b)|31$. Because $31$ is prime, we know that $o(b) = 1$ or $o(b) = 31$. Because $b \neq e$, we know that $o(b) \neq 1$, and therefore $o(b) = 31$.

9. Let

$$G = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a, b \in \mathbb{R}, \ a^2 + b^2 \neq 0 \right\}.$$ 

Show that $G$ is an abelian group with group operation matrix multiplication.
Answer: It’s easy to see that \( I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in G \). Therefore \( G \) contains the identity element.

We need to verify closure. Suppose that \( A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \) and \( B = \begin{pmatrix} c & d \\ -d & c \end{pmatrix} \). Then
\[
AB = \begin{pmatrix} ac - bd & ad + bc \\ -ad - bc & -bd + ac \end{pmatrix}.
\]
Note that the resulting matrix again has the form \( \begin{pmatrix} t & s \\ -s & t \end{pmatrix} \). Furthermore, the equation \( \det(AB) = \det(A) \det(B) \) tells us that \( \det(AB) \neq 0 \). Hence, \( AB \in G \).

Next, \( A^{-1} = \frac{1}{a^2 + b^2} \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \in G \).

Finally, to see that \( G \) is abelian, we compute \( BA = \begin{pmatrix} ca - db & cb + ad \\ -ad - bc & -bc + ac \end{pmatrix} = AB \).