Mathematics 310
Robert Gross
Homework 3
Due September 30, 2011

1. Let
$$G_1 = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \,\middle|\, a, b \in \mathbf{R},\ a^2 + b^2 \neq 0 \right\}.$$
Last week, we saw that G_1 is an abelian group with group operation matrix multiplication.

Define a function $\phi : \mathbf{C}^\times \to G_1$ with the formula $\phi(a + ib) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. Show that ϕ is an isomorphism.

2. Let
$$G_2 = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \,\middle|\, x \in \mathbf{R} \right\}.$$
(a) Show that G_2 is a group with group operation matrix multiplication.
(b) Show that the function $\phi : \mathbf{R} \to G_2$ defined by $\phi(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ is an isomorphism of the group $(\mathbf{R}, +)$ with G_2.

3. Let
$$G_3 = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \,\middle|\, a, b \in \mathbf{R}, a \neq 0 \right\}.$$
Show that G_3 is a group with group operation matrix multiplication.

4. Let
$$H_3 = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in G_3 \,\middle|\, a \in \mathbf{Q}, a \neq 0 \right\}.$$
Show that $H_3 \lhd G_3$.

5. Recall that we talked about the group
$$G_4 = \left\{ T_{a,b} : \mathbf{R} \to \mathbf{R} \,\middle|\, T_{a,b}(x) = ax + b,\ a, b \in \mathbf{R},\ a \neq 0 \right\},$$
with group operation matrix multiplication. Define $\phi : G_4 \to G_3$ with $\phi(T_{a,b}) = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$. Show that ϕ is an isomorphism.

6. Suppose that G is a group, and M and N are subgroups of G. Suppose further that $N \lhd G$. Let $MN = \left\{ mn \,\middle|\, m \in M, n \in N \right\}$. Prove that MN is a subgroup of G.

7. Suppose that G is a group, and M and N are normal subgroups of G.
(a) Show that $M \cap N$ is a normal subgroup of G.
(b) The previous problem shows that MN is a subgroup of G. Now show that MN is a normal subgroup of G.