Mathematics 310
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Homework 3
Answers

1. Let
$$G_1 = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \,\middle|\, a, b \in \mathbf{R},\ a^2 + b^2 \neq 0 \right\}.$$
Last week, we saw that G_1 is an abelian group with group operation matrix multiplication.

Define a function $\phi : \mathbf{C}^\times \to G_1$ with the formula $\phi(a+ib) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. Show that ϕ is an isomorphism.

Answer: We need to show that ϕ is a homomorphism, that ϕ is 1–1, and that ϕ is surjective.

First, we check that $\phi(a+ib)\phi(c+id) = \phi((a+ib)(c+id))$. On the left-hand side, we have
$$\begin{aligned} \phi(a+ib) &= \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \\ \phi(c+id) &= \begin{pmatrix} c & d \\ -d & c \end{pmatrix} \\ \phi(a+ib)\phi(c+id) &= \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} c & d \\ -d & c \end{pmatrix} \\ &= \begin{pmatrix} ac-bd & ad+bc \\ -bc-ad & -bd+ac \end{pmatrix} \end{aligned}$$
while
$$\begin{aligned} \phi((a+ib)(c+id)) &= \\ \phi(ac-bd+i(ad+bc)) &= \begin{pmatrix} ac-bd & ad+bc \\ -ad-bc & ac-bd \end{pmatrix} \end{aligned}$$
This shows that ϕ is a homomorphism.

Now, we can verify that ϕ is an injection just by observing that $\ker(\phi) = 1$. Finally, ϕ is onto because given any matrix $\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in G_1$, obviously $\phi(a+bi) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$.

2. Let
$$G_2 = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \,\middle|\, x \in \mathbf{R} \right\}.$$

(a) Show that G_2 is a group with group operation matrix multiplication.

(b) Show that the function $\phi : \mathbf{R} \to G_2$ defined by $\phi(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ is an isomorphism of the group $(\mathbf{R}, +)$ with G_2.

Answer: (*a*) We have

$$\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & x+y\\ 0 & 1\end{pmatrix}$$
$$\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}^{-1} = \begin{pmatrix}1 & -x\\ 0 & 1\end{pmatrix}$$

which shows that G_2 is closed under the group operation and contains inverses. Finally, setting $x = 0$ shows that $I \in G_2$.

(*b*) Now, we have

$$\phi(x)\phi(y) = \begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & x+y\\ 0 & 1\end{pmatrix} = \phi(x+y)$$

showing that ϕ is a homomorphism. It is easy to compute $\ker(\phi) = 0$, which shows that ϕ is 1–1. Finally, we observe that ϕ is onto because of the way that ϕ and G_2 are defined.

3. Let

$$G_3 = \left\{ \begin{pmatrix}a & b\\ 0 & 1\end{pmatrix} \,\middle|\, a, b \in \mathbf{R}, a \neq 0 \right\}.$$

Show that G_3 is a group with group operation matrix multiplication.

Answer: We check closure and inverses:

$$\begin{pmatrix}a & b\\ 0 & 1\end{pmatrix}\begin{pmatrix}c & d\\ 0 & 1\end{pmatrix} = \begin{pmatrix}ac & ad+b\\ 0 & 1\end{pmatrix}$$
$$\begin{pmatrix}a & b\\ 0 & 1\end{pmatrix}^{-1} = \begin{pmatrix}a^{-1} & -ba^{-1}\\ 0 & 1\end{pmatrix}$$

Setting $a = 1$ and $b = 0$ shows that $I \in G_3$.

4. Let

$$H_3 = \left\{ \begin{pmatrix}a & b\\ 0 & 1\end{pmatrix} \in G_3 \,\middle|\, a \in \mathbf{Q}, a \neq 0 \right\}.$$

Show that $H_3 \triangleleft G_3$.

Answer: If $\begin{pmatrix}a & b\\ 0 & 1\end{pmatrix}, \begin{pmatrix}c & d\\ 0 & 1\end{pmatrix} \in H_3$, then $\begin{pmatrix}a & b\\ 0 & 1\end{pmatrix}\begin{pmatrix}c & d\\ 0 & 1\end{pmatrix} = \begin{pmatrix}ac & ad+b\\ & 1\end{pmatrix} \in H_3$ because $ac \in \mathbf{Q}$. Similarly, $\begin{pmatrix}a & b\\ 0 & 1\end{pmatrix}^{-1} \in H_3$, because $a^{-1} \in \mathbf{Q}$. Finally, $I \in H_3$. This shows that H_3 is a subgroup. It remains to verify normality.

Suppose that $h = \begin{pmatrix}a & b\\ 0 & 1\end{pmatrix} \in H_3$. Let $g = \begin{pmatrix}c & d\\ 0 & 1\end{pmatrix} \in G_3$. Let $g^{-1} = \begin{pmatrix}c^{-1} & -dc^{-1}\\ 0 & 1\end{pmatrix}$.

Then $ghg^{-1} = \begin{pmatrix}c & d\\ 0 & 1\end{pmatrix}\begin{pmatrix}a & b\\ 0 & 1\end{pmatrix}\begin{pmatrix}c^{-1} & -dc^{-1}\\ 0 & 1\end{pmatrix} = \begin{pmatrix}ac & bc+d\\ 0 & 1\end{pmatrix}\begin{pmatrix}c^{-1} & -dc^{-1}\\ 0 & 1\end{pmatrix} = \begin{pmatrix}c & *\\ 0 & 1\end{pmatrix}$.

Therefore, $ghg^{-1} \in H_3$ because $c \in \mathbf{Q}$; the exact form of the entry in the upper right corner is irrelevant.

5. Recall that we talked about the group

$$G_4 = \left\{ T_{a,b} : \mathbf{R} \to \mathbf{R} \;\middle|\; T_{a,b}(x) = ax + b,\ a, b \in \mathbf{R},\ a \neq 0 \right\},$$

with group operation matrix multiplication. Define $\phi : G_4 \to G_3$ with $\phi(T_{a,b}) = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$. Show that ϕ is an isomorphism.

Answer: We have

$$\phi(T_{a,b}) = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

$$\phi(T_{c,d}) = \begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix}$$

$$\phi(T_{a,b}T_{c,d}) = \begin{pmatrix} ac & ad + b \\ 0 & 1 \end{pmatrix}$$

and $T_{a,b}T_{c,d} = T_{ac,ad+b}$. This shows that ϕ is a homomorphism.

We compute $\ker(\phi) = T_{1,0}$, which is the identity element in G_4. Therefore, ϕ is 1–1. Finally, again the definition of ϕ and G_3 make is clear that ϕ is surjective.

6. Suppose that G is a group, and M and N are subgroups of G. Suppose further that $N \triangleleft G$. Let $MN = \left\{ mn \;\middle|\; m \in M, n \in N \right\}$. Prove that MN is a subgroup of G.

Answer: We know that $e \in M$ and $e \in N$, so $e = e \cdot e \in MN$.

Next, let's take care of inverses. Suppose that $mn \in MN$. Then $(mn)^{-1} = n^{-1}m^{-1} = (m^{-1}m)(n^{-1}m^{-1}) = m^{-1}(mn^{-1}m^{-1})$. Now, we know that $m^{-1} \in M$, and we have $mn^{-1}m^{-1} \in N$ because $n^{-1} \in N$ and $N \triangleleft G$. So $m^{-1}(mn^{-1}m^{-1}) \in MN$.

Finally, what about closure? Suppose that $m_1n_1, m_2n_2 \in MN$. We need to verify that $m_1n_1m_2n_2 \in MN$. The idea is another trick: $(m_1n_1)(m_2n_2) = (m_1m_2)(m_2^{-1}n_1m_2)n_2$. Now, $m_1m_2 \in M$. Because of normality, $m_2^{-1}n_1m_2 \in N$, and $n_2 \in N$, so $(m_2^{-1}n_1m_2)n_2 \in N$. Therefore, $m_1n_1m_2n_2 \in MN$, proving that MN is a subgroup.

7. Suppose that G is a group, and M and N are normal subgroups of G.

   (a) Show that $M \cap N$ is a normal subgroup of G.

   (b) The previous problem shows that MN is a subgroup of G. Now show that MN is a normal subgroup of G.

Answer: (a) We showed in the first homework that if A and B are subgroups of G, so is $A \cap B$. Now we need to show that $M \cap N$ is normal.

Take $k \in M \cap N$, and $g \in G$. We must show that $gkg^{-1} \in M \cap N$. Because $k \in M \cap N$, we know that $k \in M$. Because $M \triangleleft G$, we know that $gkg^{-1} \in M$. Similarly, we know that $gkg^{-1} \in N$. Therefore, $gkg^{-1} \in M \cap N$, showing that $M \cap N$ is a normal subgroup.

(b) Suppose that $mn \in MN$. Let $g \in G$. We need to show that $g(mn)g^{-1} \in MN$. Write $g(mn)g^{-1} = (gmg^{-1})(gng^{-1})$. Then $gmg^{-1} \in M$, because $M \triangleleft G$, and $gng^{-1} \in N$, because $N \triangleleft G$. Therefore, $(gmg^{-1})(gng^{-1}) \in MN$, showing that $MN \triangleleft G$.