Mathematics 310
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Homework 4
Due October 14, 2011

1. Suppose that m, n, and k are positive integers, with $(m,n)=1$, $m|k$, and $n|k$. Prove that $mn|k$.

2. Suppose that G is an abelian group, with $a,b \in G$. Suppose that $o(a)=m$ and $o(b)=n$, and $(m,n)=1$. Prove that $o(ab)=mn$. *Note:* It is clear that $(ab)^{mn}=e$; the point is that you must show that no smaller exponent j satisfies $(ab)^j=e$.

3. Suppose that G is a finite abelian group, and $o(G)=p^a m$, where $p \nmid m$, $a \geq 1$, and p is a prime. Let $H=\{g \in G : g^{p^a}=e\}$.
 (a) Prove that H is a subgroup of G.
 (b) Prove that if $h \in H$, then the only prime that might divide $o(h)$ is p.
 (c) Prove that the only prime dividing $o(H)$ is p. *Hint:* Apply Cauchy's Theorem.
 (d) Show that $p \nmid o(G/H)$. *Hint:* Cauchy's Theorem says that if $p|o(G/H)$, then G/H contains a coset of order p. Now use an argument similar to the one which we used to prove Cauchy's Theorem.
 (e) Show that $o(H)=p^a$.

This is a specific case of one of the Sylow Theorems, which apply to both abelian and non-abelian groups. The proof is much trickier in the case of non-abelian groups.

4. If $\phi : G_1 \to G_2$ is a surjective homomorphism, and $N \lhd G_1$, show that $\phi(N) \lhd G_2$. You may assume that $\phi(N)$ is a subgroup of G_2.

5. If H is any subgroup of G, let $N(H)$ be defined by:

$$N(H)=\{a \in G \mid aH=Ha\}.$$

Prove that:
 (a) $N(H)$ is a subgroup of G, and $N(H) \supset H$.
 (b) $H \lhd N(H)$.
 (c) If K is a subgroup of G such that $H \lhd K$, then $K \subset N(H)$.

These facts combine to tell us that $N(H)$ is the largest subgroup of G in which H is normal. The group $N(H)$ is called the *normalizer* of H.