Mathematics 310
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Homework 5
Due October 21, 2011

1. Suppose that G is a finite group, with H a subgroup of G and g an element of G. Suppose that k is the smallest positive integer so that $g^k \in H$. Prove that $k | o(g)$. *Hint*: Use the division algorithm to write $o(g) = qk + r$, and show that $r = 0$.

2. Suppose that G is a group, H a subgroup of G, and $N \triangleleft G$. Show that $H \cap N \triangleleft H$.

3. Suppose that $\phi : G_1 \to G_2$ is a homomorphism, and $N_2 \triangleleft G_2$. Let

$$N_1 = \{g_1 \in G_1 : \phi(g_1) \in N_2\}.$$

(1) Show that $N_1 \triangleleft G_1$.
(2) Show that $\ker(\phi) \subset N_1$.

4. Suppose that G is a finite group with subgroups A and B. Suppose that $o(A) > o(B) > \sqrt{o(G)}$. Prove that $A \cap B \neq \{e\}$.

5. Suppose that G_1 is a group, and $G = G_1 \times G_1$. Let $D = \{(a, a) \in G : a \in G_1\}$.
(1) Show that D is a subgroup of G.
(2) Suppose that $D \triangleleft G$. Prove that G_1 is abelian.

6. If $M \triangleleft G$, $N \triangleleft G$, and $M \cap N = \{e\}$, show that for $m \in M$, $n \in N$, $mn = nm$. *Hint*: Show that $mnm^{-1}n^{-1} \in M \cap N$.

7. Recall that an *automorphism* of a group G is an isomorphism $\phi : G \to G$. Find all automorphisms of the group $\mathbf{Z}/8\mathbf{Z}$. To define an automorphism, you can show explicitly where it maps each of the 8 elements of $\mathbf{Z}/8\mathbf{Z}$. For example, the identity isomorphism is

$$\text{id:}\quad \begin{array}{c} 0 \to 0 \\ 1 \to 1 \\ 2 \to 2 \\ 3 \to 3 \\ 4 \to 4 \\ 5 \to 5 \\ 6 \to 6 \\ 7 \to 7 \end{array}$$