Mathematics 310
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Homework 5
Answers

1. Suppose that G is a finite group, with H a subgroup of G and g an element of G. Suppose that k is the smallest positive integer so that $g^k \in H$. Prove that $k | o(g)$. *Hint*: Use the division algorithm to write $o(g) = qk + r$, and show that $r = 0$.

Answer: Write $o(g) = qk+r$, with $0 \le r < k$. We have $g^{o(g)} = e \in H$, and $g^k \in H$. Therefore, $e = (g^k)^q g^r \in H$, so $g^r = (g^k)^{-q} \in H$. But $r < k$, and k is the smallest positive integer so that $g^k \in H$. The conclusion is that $r = 0$ and then $k | o(g)$.

2. Suppose that G is a group, H a subgroup of G, and $N \triangleleft G$. Show that $H \cap N \triangleleft H$.

Answer: We know that $H \cap N$ is a subgroup from earlier homework, so the only issue is proving normality. Suppose that $h \in H$ and $n \in H \cap N$. We must show that $hnh^{-1} \in H \cap N$. Because $h \in H$ and $n \in H$, we know that $hnh^{-1} \in H$. Our remaining problem is to show that $hnh^{-1} \in N$, but that follows because $N \triangleleft G$ and $h \in G$.

3. Suppose that $\phi : G_1 \to G_2$ is a homomorphism, and $N_2 \triangleleft G_2$. Let

$$N_1 = \{g_1 \in G_1 : \phi(g_1) \in N_2\}.$$

(1) Show that $N_1 \triangleleft G_1$.
(2) Show that $\ker(\phi) \subset N_1$.

Answer: (1) We showed on the examination that N_1 is a subgroup of G_1, so normality is the only remaining problem. Take $g_1 \in G_1$ and $n_1 \in N_1$, so that $\phi(n_1) = n_2 \in N_2$, and we must show that $g_1 n_1 g_1^{-1} \in N_1$. We have $\phi(g_1 n_1 g_1^{-1}) = \phi(g_1)\phi(n_1)\phi(g_1)^{-1} = \phi(g_1) n_2 \phi(g_1)^{-1} \in N_2$. Therefore, $g_1 n_1 g_1^{-1} \in N_1$.

(2) If $k \in \ker(\phi)$, then $\phi(k) = e_2 \in N_2$, and therefore $k \in N_1$.

4. Suppose that G is a finite group with subgroups A and B. Suppose that $o(A) > o(B) > \sqrt{o(G)}$. Prove that $A \cap B \neq \{e\}$.

Answer: Suppose that $A \cap B = \{e\}$. We compute the number of elements in $AB = \{ab \mid a \in A, b \in B\}$. If $a_1 b_1 = a_2 b_2$, then $a_2^{-1} a_1 = b_2 b_1^{-1} \in A \cap B = \{e\}$, implying that $a_1 = a_2$ and $b_1 = b_2$. This shows that $o(AB) = o(A)o(B) > \sqrt{o(G)}\sqrt{o(G)} = G$. But $AB \subseteq G$, so we have a contradiction. The conclusion is that $A \cap B \neq \{e\}$.

5. Suppose that G_1 is a group, and $G = G_1 \times G_1$. Let $D = \{(a, a) \in G : a \in G_1\}$.

(1) Show that D is a subgroup of G.
(2) Suppose that $D \triangleleft G$. Prove that G_1 is abelian.

Answer: (1) First, $(e, e) \in D$, so D contains the identity element in G.

Second, if $(a, a), (b, b) \in D$, then $(a, a)(b, b) = (ab, ab) \in D$.

Third, if $(a, a) \in D$, then $(a, a)^{-1} = (a^{-1}, a^{-1}) \in D$.

This shows that D is a subgroup.

(2) Suppose that $D \triangleleft G$. Take two elements $a, b \in G_1$, and we must show that $ab = ba$. The element $(a, a) \in D$, and the element $(e, b) \in G$, and therefore $(e, b)(a, a)(e, b)^{-1} \in D$. We

compute $(e,b)(a,a)(e,b)^{-1} = (a, bab^{-1})$. But if $(a, bab^{-1}) \in D$, then $a = bab^{-1}$, or $ab = ba$, which proves that G_1 is abelian.

6. If $M \triangleleft G$, $N \triangleleft G$, and $M \cap N = \{e\}$, show that for $m \in M$, $n \in N$, $mn = nm$. *Hint*: Show that $mnm^{-1}n^{-1} \in M \cap N$.

Answer: Note that $mnm^{-1}n^{-1} = (mnm^{-1})n^{-1}$. Because $N \triangleleft G$, we have $mnm^{-1} \in N$, and therefore $(mnm^{-1})n^{-1} \in N$.

Note as well that $mnm^{-1}n^{-1} = m(nm^{-1}n^{-1})$. Because $M \triangleleft G$, we have $nm^{-1}n^{-1} \in M$ (because $m^{-1} \in M$). Therefore, $m(nm^{-1}n^{-1}) \in M$.

So $mnm^{-1}n^{-1} \in M \cap N = \{e\}$, so $mnm^{-1}n^{-1} = e$, which in turn says that $mn = nm$.

7. Recall that an *automorphism* of a group G is an isomorphism $\phi : G \to G$. Find all automorphisms of the group $\mathbf{Z}/8\mathbf{Z}$. To define an automorphism, you can show explicitly where it maps each of the 8 elements of $\mathbf{Z}/8\mathbf{Z}$. For example, the identity isomorphism is

$$\text{id:}\quad \begin{array}{l} 0 \to 0 \\ 1 \to 1 \\ 2 \to 2 \\ 3 \to 3 \\ 4 \to 4 \\ 5 \to 5 \\ 6 \to 6 \\ 7 \to 7 \end{array}$$

Answer: If $\phi : \mathbf{Z}/8\mathbf{Z} \to \mathbf{Z}/8\mathbf{Z}$ is a homomorphism, we know that $\phi(0) = 0$, so we only have to define ϕ for the remaining 7 elements of $\mathbf{Z}/8\mathbf{Z}$. We also know that if ϕ is an injection and $x \neq 0$, then $\phi(x) \neq 0$, because $\phi(0) = 0$.

Now, if $\phi(1) = 2$, then ϕ is not an injection, because $\phi(4) = \phi(1+1+1+1) = \phi(1)+\phi(1)+\phi(1)+\phi(1) = 2+2+2+2 = 0$. Similarly, if $\phi(1) = 4$, then ϕ is not an injection, because $\phi(2) = \phi(1+1) = \phi(1)+\phi(1) = 4+4 = 0$. Finally, if $\phi(1) = 6$, then $\phi(4) = \phi(1+1+1+1) = \phi(1)+\phi(1)+\phi(1)+\phi(1) = 6+6+6+6 = 0$.

The remaining possibilities are $\phi_3(1) = 3$, meaning that $\phi_3(x) = 3x$; $\phi_5(1) = 5$, meaning $\phi_5(x) = 5x$; and $\phi_7(1) = 7$, meaning $\phi_7(x) = 7x$. Because $\phi_c(x+y) = c(x+y) = cx + cy = \phi_c(x) + \phi_c(y)$, we know that these three functions are homomorphisms. Because $\phi_3 \circ \phi_3 = \phi_5 \circ \phi_5 = \phi_7 \circ \phi_7 = \text{id}$, we know that each of these three functions is invertible—in fact, each is its own inverse—and therefore each must be a bijection.