Mathematics 310
Robert Gross
Homework 6
Answers

1. Suppose that G is a finite group, H a subgroup, and $o(H) = o(G)/2$. Prove that $H \triangleleft G$. *Hint*: The best approach is to show that all left cosets are right cosets, and *vice versa*.

Answer: Pick any element $g \in G \setminus H$. Then the two right cosets of H are H and Hg. We know that $H \cup Hg = G$ and $H \cap Hg = \emptyset$. Similarly, the two left cosets of H are H and gH, and we know that $H \cup gH = G$ and $H \cap gH = \emptyset$.

These statements imply that $gH = Hg$: If $x \in gH$, then $x \in G \setminus H$, and therefore $x \in Hg$; similarly, we can start with $x \in Hg$ and conclude $x \in gH$.

If every left coset of H equals a right coset of H, then H is normal.

Hint: Remember that we proved that if $\phi : G_1 \to G_2$ is a homomorphism of finite groups G_1 to G_2, and $a \in G_1$, we proved that $o(\phi(a))|o(a)$. That fact might be useful in the next few problems.

2. Show that the only homomorphism from $\mathbf{Z}/5\mathbf{Z}$ to $\mathbf{Z}/8\mathbf{Z}$ is the trivial homomorphism

Answer: Suppose that $\phi : \mathbf{Z}/5\mathbf{Z} \to \mathbf{Z}/8\mathbf{Z}$. Pick any $n \in \mathbf{Z}/5\mathbf{Z}$. If $n = 0$, then we know that $\phi(n) = 0$, because $\phi(e_1) = e_2$. Suppose that $n \neq 0$. Then $o(n) = 5$. We know that $o(\phi(n))|o(n)$, so $o(\phi(n))$ must be 1 or 5. But we also know that $o(\phi(n))|o(\mathbf{Z}/8\mathbf{Z}) = 8$, and that implies that $o(\phi(n)) = 1$. This is a fancy way of writing $\phi(n) = 0$, and therefore the homomorphism is trivial.

3. Show that the only homomorphism from S_3 to $\mathbf{Z}/3\mathbf{Z}$ is the trivial homomorphism.

Answer: Suppose that $\phi : S_3 \to \mathbf{Z}/3\mathbf{Z}$ is a homomorphism. Consider $\phi(12)$. We know that the order of (12) is 2, and therefore $o(\phi(12))|2$. But we also know that $\phi(12) \in \mathbf{Z}/3\mathbf{Z}$, and therefore $o(\phi(12))|3$. The conclusion must be that $o(\phi(12)) = 1$, and therefore $\phi(12) = e$. Similarly, $\phi(13) = \phi(23) = e$, so the kernel of ϕ contains e, (12), (23), and (13). Because the kernel is a subgroup of S_3, its order must divide 6. We now have at least 4 elements in the kernel, and we can conclude that the kernel must contain 6 elements; therefore $\phi(\tau) = 0$ for every $\tau \in S_3$.

4. Let G_1 be a finite group, and let $\phi : G_1 \to G_2$ be an injective homomorphism. Let a be any element of G_1. Show that $o(\phi(a)) = o(a)$.

Answer: Let $n = o(a)$ and $m = o(\phi(a))$. We know that $\phi(a)^n = \phi(a^n) = \phi(e_1) = e_2$, and therefore $m|n$.

On the other hand, we know that $\phi(a^m) = \phi(a)^m = e_2$. Because ϕ is an injection, this implies that $a^m = e_1$, and therefore $n|m$. If $n|m$ and $m|n$, we can conclude that $m = n$.

5. Find a non-trivial homomorphism from $\mathbf{Z}/3\mathbf{Z}$ to S_3.

Answer: Suppose that $\phi : \mathbf{Z}/3\mathbf{Z} \to S_3$ is a homomorphism. We must have $\phi(0) = e$. We know that $o(1) = 3$, and therefore $o(\phi(1))|3$. That means that $o(\phi(1)) = 1$, in which case the homomorphism is trivial, or $o(\phi(1)) = 3$. In that case, there are two possibilities. We could have $\phi(1) = (123)$, and then $\phi(2) = \phi(1 + 1) = (123)(123) = (132)$. Or we could have $\phi(1) = (132)$, and then $\phi(2) = \phi(1+1) = (132)(132) = (123)$. Those are the only two non-trivial homomorphisms.

6. Find the cycle decomposition and order.

(a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 1 & 4 & 2 & 7 & 9 & 6 & 8 & 5 \end{pmatrix}$.

(b) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 6 & 5 & 3 & 4 & 2 & 1 \end{pmatrix}$.

(c) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 6 & 5 & 3 & 4 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 1 & 5 & 6 & 7 & 4 \end{pmatrix}$.

Classify each of these 3 permutations as even or odd.

Answer: (a) We have $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 1 & 4 & 2 & 7 & 9 & 6 & 8 & 5 \end{pmatrix} = (1342)(5769)$, and therefore this permutation has order 4. We also have $(1342)(5769) = (12)(14)(13)(59)(56)(57)$, a product of 6 transpositions. Because 6 is an even number, this is an even transposition.

(b) We have $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 6 & 5 & 3 & 4 & 2 & 1 \end{pmatrix} = (17)(26)(354) = (17)(26)(34)(35)$. We see that this permutation has order 6, and that it is also even, because it is a product of 4 transpositions.

(c) We have $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 6 & 5 & 3 & 4 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 1 & 5 & 6 & 7 & 4 \end{pmatrix} = (16)(25)(37)$. This computation shows that the order of the permutation is 2, and that this is an odd permutation.