Mathematics 310
Robert Gross
Homework 7
Due November 4, 2011

1. Suppose that G is a finite group with subgroups A and B. Prove that $o(AB) = o(A)o(B)/o(A \cap B)$. Note that typically, AB will just be a subset of G and not a subgroup.

2. If $(m, n) = 1$, show that the only group homomorphism $\phi : \mathbf{Z}/m\mathbf{Z} \to \mathbf{Z}/n\mathbf{Z}$ is the trivial homomorphism. Remember that the group operation is addition.

3. Find a non-trivial group homomorphism $\phi : \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z} \to \mathbf{Z}/4\mathbf{Z}$.

4. Find a non-trivial group homomorphism $\phi : \mathbf{Z}/4\mathbf{Z} \to \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$.

5. Remember that the Hamiltonians $\mathbf{H}$ are defined by $\mathbf{H} = \{x_1 + ix_2 + jx_3 + kx_4 : x_1, x_2, x_3, x_4 \in \mathbf{R}\}$ with $ij = k$, $jk = i$, $ki = j$, and $i^2 = j^2 = k^2 = -1$. Show there are infinitely many elements $x \in \mathbf{H}$ satisfying $x^2 = -1$.

6. If R, S are rings, define the *direct sum* of R and S, $R \oplus S$, by

$$R \oplus S = \{(r, s) : r \in R, s \in S\}$$

where $(r, s) = (r_1, s_1)$ if and only if $r = r_1$ and $s = s_1$, and where we define

$$(r, s) + (t, u) = (r + t, s + u), \quad (r, s)(t, u) = (rt, su).$$

(*a*) Show that $R \oplus S$ is a ring.
(*b*) Show that $\{(r, 0) : r \in R\}$ and $\{(0, s) : s \in S\}$ are ideals of $R \oplus S$.
(*c*) Show that $\mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/3\mathbf{Z}$ is ring isomorphic to $\mathbf{Z}/6\mathbf{Z}$.
(*d*) Show that $\mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$ is not ring isomorphic to $\mathbf{Z}/4\mathbf{Z}$.