Mathematics 310
Robert Gross
Homework 9
Due November 28, 2011

1. Suppose that F is a field. Let

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \;\middle|\; a, b, c \in F \right\}$$
$$I = \left\{ \begin{pmatrix} 0 & d \\ 0 & 0 \end{pmatrix} \;\middle|\; d \in F \right\}$$

Show that

(a) R is a ring.

(b) I is an ideal of R.

(c) The function $\phi : R \to F \oplus F$ defined by $\phi\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = (a, c)$ is a ring homomorphism with kernel I.

2. Let p be a prime. Show that the polynomial $x^{p-1} + x^{p-2} + \cdots + x + 1$ is irreducible in $\mathbf{Q}[x]$.

3. Suppose that K and L are two fields, with $K \subset L$. Suppose that $\dim_K(L) = n$. Let $a \in K$. Show that there are elements α_0, α_1, $\ldots$, α_n of K, not all zero, so that $\sum_{k=0}^{n} \alpha_k a^k = 0$.

4. Let F be a field, let $f(x) \in F[x]$ be an irreducible polynomial, and suppose $\deg(f) = n \geq 1$. Let $M = (f(x))$, and let $K = F[x]/M$. We know that K is a field containing F. Show that $\dim_F(K) = n$.

5. Suppose that F is a field, R is a ring, and $\phi : F \to R$ is a surjective ring homomorphism. Show that ϕ is a bijection, and that R is a field.

6. Show that $\mathbf{R} \oplus \mathbf{R}$ is *not* ring isomorphic to $\mathbf{C}$.