Mathematics 310
Robert Gross
Homework 10
Answers

1. Show that there is no ring homomorphism $\phi : \mathbf{C} \to \mathbf{R} \oplus \mathbf{R}$.

Answer: Suppose that there is such a homomorphism ϕ. Then $\phi(1) = (1, 1)$, by our definition of ring homomorphism, and therefore $\phi(-1) = (-1, -1)$. Suppose that $\phi(i) = (a, b)$. Then $(a^2, b^2) = \phi(i^2) = \phi(-1) = (-1, -1)$, forcing $a^2 = b^2 = -1$. Because that is not possible if $a, b \in \mathbf{R}$, we know that there is no homomorphism.

2. Find a ring homomorphism $\phi : \mathbf{R} \oplus \mathbf{R} \to \mathbf{C}$.

Answer: One possibility is $\phi(a, b) = a$; another is $\phi(a, b) = b$.

3. Find the minimal polynomial in $\mathbf{Q}[x]$ for $\sqrt{2} + \sqrt[3]{7}$.

Answer: Let $x = \sqrt{2}+\sqrt[3]{7}$. Then $x-\sqrt{2} = \sqrt[3]{7}$, and so $7 = (x-\sqrt{2})^3 = x^3-3\sqrt{2}x^2+6x-2\sqrt{2}$. Rearranging, we get $x^3+6x-7 = \sqrt{2}(3x^2+2)$. Now, squaring gives $x^6+36x^2+49+12x^4-14x^3-84x = 2(9x^4+12x^2+4) = 18x^4+24x^2+8$. Rearranging yields $x^6-6x^4-14x^3+12x^2-84x+41 = 0$.

4. Suppose that E is a field containing q elements, and $E \subset F$. Suppose that F is a field, with $[F : E] = n$. Show that F contains q^n elements.

Answer: If $[F : E] = n$, then there is a basis $\{x_1, \ldots, x_n\}$ of F over E. In other words, every element of F can be uniquely written in the form $e_1x_1 + \cdots + e_nx_n$, where $e_1, e_2, \ldots, e_n \in E$. Because there are q possibilities for each of the elements $e_1, \ldots, e_n$, we know that there are q^n elements in F.

5. Suppose that $E \subset F$, where E and F are fields, and suppose as well that $[E : F] = p$, where p is a prime. Let a be any element of $F \setminus E$. Show that $F = E(a)$.

Answer: If a is an element of F which is not in E, then the field $E(a)$ is a field which contains E but does not equal E. We have $E \subset E(a) \subset F$, and therefore $p = [F : E] = [F : E(a)][E(a) : E]$. However, primes have a limited number of factors. Because $[E(a) : E] \neq 1$, we know that $[E(a) : E] = p$, and therefore $[F : E(a)] = 1$, meaning that $F = E(a)$.

6. Degrees do not always behave the way that we would hope. Find two numbers a and b which are algebraic over $\mathbf{Q}$ with $[\mathbf{Q}(a) : \mathbf{Q}] = 2$, $[\mathbf{Q}(b) : \mathbf{Q}] = 3$, but the degree of the minimal polynomial for ab is *less* than 6.

Answer: This is hard to find without some thought. The trick is to make ab solve the same minimal polynomial as b. So one possibility is to take $b = \sqrt[3]{2}$, with minimal polynomial $x^3 - 2$. The other roots of this polynomial are ab and a^2b, where $a = \frac{-1+\sqrt{-3}}{2}$. The minimal polynomial for a is $x^2 + x + 1 = 0$, meaning that $[\mathbf{Q}(a) : \mathbf{Q}] = 2$, $[\mathbf{Q}(\sqrt[3]{2}) : \mathbf{Q}] = 3$, and $[\mathbf{Q}(a\sqrt[3]{2}) : \mathbf{Q}] = 3$.