Mathematics 414
Exam 1
Answers

1. *(20 points)* Let $f(x) = x^2 - x - 1$. Let $g(x) = \sqrt{x+1}$.
   (a) Show that if p is a fixed point of g, then p is a root of f.
   (b) Show that if p is a positive root of f, then p is a fixed point of g.
   (c) Let $p_0 = 1.0$, and define $p_n = g(p_{n-1})$. Compute p_1 and p_2.
   (d) Using the values of p_0, p_1, and p_2 that you just computed, compute $\hat{p}_0$ using the Aitken's Δ^2-method.
   (e) Perform two iterations of Newton's method with $p_0 = 1$.
   (f) Let $p_0 = 1.0$, $p_1 = 1.1$, and $p_2 = 1.2$. Using Müller's Method to compute p_3.
   (g) Explain why Müller's method gives a better answer than either of the other methods.

Answer: (a) If $g(p) = p$, then $p = \sqrt{p+1}$, so $p^2 = p + 1$, and then $p^2 - p - 1 = 0$.

(b) If $f(p) = 0$, then $p^2 = p - 1$, and we can take square roots of both sides provided that $p \geq 0$, and conclude that $p = g(p)$.

(c) I compute that $p_1 = 1.414214$ and $p_2 = 1.553774$.

(d) I compute $\hat{p}_0 = 1.624689$.

(e) Now, I compute that $p_1 = 2$ and $p_2 = 1.666667$.

(f) I compute that $a = 1.0$, $b = 1.4$, $c = -0.76$, and $p3 = 1.618034$.

(g) We are solving a quadratic polynomial, and Müller's method involves computing the point of intersection of a quadratic polynomial and the x-axis. Usually, the quadratic polynomial involved is an approximation to $f(x)$, but in this case it is exactly $f(x)$, and so Müller's method gives the exact answer.

2. *(20 points)* Suppose that $f_1(x)$ has a zero of multiplicity m_1 at a, and $f_2(x)$ has a zero of multiplicity m_2 at a, with $m_1 < m_2$.
   (a) Let $g(x) = f_1(x)f_2(x)$. Show that $g(x)$ has a zero of multiplicity $m_1 + m_2$ at a.
   (b) Let $h(x) = f_1(x) + f_2(x)$. Show that $h(x)$ has a zero of multiplicity m_1 at a.

Answer: Write $f_1(x) = (x-a)^{m_1} q_1(x)$ and $f_2(x) = (x-a)^{m_2} q_2(x)$, with $q_1(a) \neq 0$ and $q_2(a) \neq 0$.

(a) We have $g(x) = (x-a)^{m_1} q_1(x)(x-a)^{m_2} q_2(x) = (x-a)^{m_1+m_2} q_1(x)q_2(x)$. We know that $q_1(a)q_2(a) \neq 0$, so this shows that $g(x)$ has a zero of multiplicity $m_1 + m_2$.

(b) We have $h(x) = (x-a)^{m_1} q_1(x) + (x-a)^{m_2} q_2(x) = (x-a)^{m_1}(q_1(x) + (x-a)^{m_2-m_1} q_2(x))$. Let $q_3(x) = q_1(x) + (x-a)^{m_2-m_1} q_2(x)$, so we have $h(x) = (x-a)^{m_1} q_3(x)$. We see that $q_3(a) = q_1(a) \neq 0$, and therefore $h(x)$ has a zero of multiplicity m_1.

3. *(20 points)* Perform the following two computations:

$$\left(3 + \frac{1}{13}\right) + \frac{1}{19} \qquad 3 + \left(\frac{1}{13} + \frac{1}{19}\right)$$

(a) using 3-digit rounding arithmetic.

(b) using 3-digit chopping arithmetic.

Answer: (a) I compute that $3+\frac{1}{13} \approx 3.08$, and therefore $(3+\frac{1}{13})+\frac{1}{19} \approx 3.08+0.0526 \approx 3.13$.
On the other hand, $\frac{1}{13}+\frac{1}{19} \approx 0.13$, and so $3+(\frac{1}{13}+\frac{1}{19}) \approx 3.13$
(b) I compute that $3+\frac{1}{13} \approx 3.07$, and therefore $(3+\frac{1}{13})+\frac{1}{19} \approx 3.07+0.0526 \approx 3.12$.
On the other hand, $\frac{1}{13}+\frac{1}{19} \approx 0.129$, and therefore $3+(\frac{1}{13}+\frac{1}{19}) \approx 3.12$.

4. *(20 points)* Let $f(x) = e^x \sin x$.
   (a) Compute $P_2(x)$, the second Taylor polynomial for $f(x)$, at $x = 0$.
   (b) Approximate $\int_0^{0.2} e^x \sin x\, dx$ by computing $\int_0^{0.2} P_2(x)\, dx$.
   (c) Use the error term $R_2(x)$ to give a good upper bound for the error in the approximation in part (b).

Answer: (a) We start by computing $f'(x) = e^x(\sin x + \cos x)$, and then $f''(x) = e^x(\sin x + \cos x) + e^x(\cos x - \sin x) = 2e^x \cos x$, and then finally $f^{(3)}(x) = 2e^x(\cos x - \sin x)$. Therefore, $f(0) = 0$, $f'(0) = 1$, and $f''(0) = 2$.

We then compute $P_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 = x + x^2$.

(b) We now approximate

$$\int_0^{0.2} e^x \sin x\, dx \approx \int_0^{0.2} (x + x^2)\, dx = \left.\frac{x^2}{2} + \frac{x^3}{3}\right]_0^{0.2} \approx 0.022667.$$

(c) We know that $R_2(x) = \frac{f^{(3)}(\xi)}{6}x^3$, where ξ is between 0 and x. In this case, $0 \le x \le 0.2$, so $0 \le \xi \le 0.2$. We know that $f^{(3)}(\xi) = 2e^{\xi}(\cos\xi - \sin\xi)$. The largest value of $\cos\xi - \sin\xi$ on this interval occurs when $\xi = 0$, and that value is 1. The largest value of e^{ξ} occurs when $\xi = 0.2$, and that value is 1.221403. So the largest value of $R_2(x)$ is $\frac{1.221403}{6} \cdot 0.2^3 \approx 0.001629$.

The error incurred in making the approximation is $\int_0^{0.2} R_2(x)\, dx \le \int_0^{0.2} 0.001629\, dx = 0.000326$. That is an upper bound for the absolute value of the error.

In this case, we can compute the integral exactly, and find that $\int_0^{0.2} e^x \sin x\, dx \approx 0.0228$.

5. *(20 points)* Let a be a fixed positive real number. Define the sequence x_n by

$$x_0 = 1 \qquad x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right), \quad n = 0, 1, 2, \ldots$$

Show that the sequence $\{x_n\}$ is the same sequence that arises by using Newton's method on the function $f(x) = x^2 - a$ with $p_0 = 1$.

Answer: Newton's method says that $p_{n+1} = p_n - f(p_n)/f'(p_n)$. In this case, that gives

$$p_{n+1} = p_n - \frac{p_n^2 - a}{2p_n} = \frac{2p_n^2 - (p_n^2 - a)}{2p_n} = \frac{p_n^2 + a}{2p_n} = \frac{1}{2}\left(\frac{p_n^2 + a}{p_n}\right) = \frac{1}{2}\left(p_n + \frac{a}{p_n}\right).$$