1. (20 points) Let \( f(x) = x^2 - x - 1 \). Let \( g(x) = \sqrt{x + 1} \).
   (a) Show that if \( p \) is a fixed point of \( g \), then \( p \) is a root of \( f \).
   (b) Show that if \( p \) is a positive root of \( f \), then \( p \) is a fixed point of \( g \).
   (c) Let \( p_0 = 1.0 \), and define \( p_n = g(p_{n-1}) \). Compute \( p_1 \) and \( p_2 \).
   (d) Using the values of \( p_0 \), \( p_1 \), and \( p_2 \) that you just computed, compute \( \hat{p}_0 \) using the Aitken’s \( \Delta^2 \)-method.
   (e) Perform two iterations of Newton’s method with \( p_0 = 1 \).
   (f) Let \( p_0 = 1.0 \), \( p_1 = 1.1 \), and \( p_2 = 1.2 \). Using Müller’s Method to compute \( p_3 \).
   (g) Explain why Müller’s method gives a better answer than either of the other methods.

   **Answer:**
   (a) If \( g(p) = p \), then \( p = \sqrt{p + 1} \), so \( p^2 = p + 1 \), and then \( p^2 - p - 1 = 0 \).
   (b) If \( f(p) = 0 \), then \( p^2 = p - 1 \), and we can take square roots of both sides provided that \( p \geq 0 \), and conclude that \( p = g(p) \).
   (c) I compute that \( p_1 = 1.414214 \) and \( p_2 = 1.553774 \).
   (d) I compute \( \hat{p}_0 = 1.624689 \).
   (e) Now, I compute that \( p_1 = 2 \) and \( p_2 = 1.666667 \).
   (f) I compute that \( a = 1.0 \), \( b = 1.4 \), \( c = -0.76 \), and \( p_3 = 1.618034 \).
   (g) We are solving a quadratic polynomial, and Müller’s method involves computing the point of intersection of a quadratic polynomial and the \( x \)-axis. Usually, the quadratic polynomial involved is an approximation to \( f(x) \), but in this case it is exactly \( f(x) \), and so Müller’s method gives the exact answer.

2. (20 points) Suppose that \( f_1(x) \) has a zero of multiplicity \( m_1 \) at \( a \), and \( f_2(x) \) has a zero of multiplicity \( m_2 \) at \( a \), with \( m_1 < m_2 \).
   (a) Let \( g(x) = f_1(x)f_2(x) \). Show that \( g(x) \) has a zero of multiplicity \( m_1 + m_2 \) at \( a \).
   (b) Let \( h(x) = f_1(x) + f_2(x) \). Show that \( h(x) \) has a zero of multiplicity \( m_1 \) at \( a \).

   **Answer:** Write \( f_1(x) = (x-a)^{m_1}q_1(x) \) and \( f_2(x) = (x-a)^{m_2}q_2(x) \), with \( q_1(a) \neq 0 \) and \( q_2(a) \neq 0 \).
   (a) We have \( g(x) = (x-a)^{m_1}q_1(x)(x-a)^{m_2}q_2(x) = (x-a)^{m_1+m_2}q_1(x)q_2(x) \). We know that \( q_1(a)q_2(a) \neq 0 \), so this shows that \( g(x) \) has a zero of multiplicity \( m_1 + m_2 \).
   (b) We have \( h(x) = (x-a)^{m_1}q_1(x) + (x-a)^{m_2}q_2(x) = (x-a)^{m_1}(q_1(x) + (x-a)^{m_2-m_1}q_2(x)) \). Let \( q_3(x) = q_1(x) + (x-a)^{m_2-m_1}q_2(x) \), so we have \( h(x) = (x-a)^{m_1}q_3(x) \). We see that \( q_3(a) = q_1(a) \neq 0 \), and therefore \( h(x) \) has a zero of multiplicity \( m_1 \).

3. (20 points) Perform the following two computations:

\[
\left(3 + \frac{1}{13}\right) + \frac{1}{19} \quad 3 + \left(\frac{1}{13} + \frac{1}{19}\right)
\]

   (a) using 3-digit rounding arithmetic.
(b) using 3-digit chopping arithmetic.

Answer: (a) I compute that $3 + \frac{1}{13} \approx 3.08$, and therefore $(3 + \frac{1}{13}) + \frac{1}{19} \approx 3.08 + 0.0526 \approx 3.13$.

On the other hand, $\frac{1}{13} + \frac{1}{19} \approx 0.13$, and so $3 + (\frac{1}{13} + \frac{1}{19}) \approx 3.13$

(b) I compute that $3 + \frac{1}{13} \approx 3.07$, and therefore $(3 + \frac{1}{13}) + \frac{1}{19} \approx 3.07 + 0.0526 \approx 3.12$.

On the other hand, $\frac{1}{13} + \frac{1}{19} \approx 0.129$, and therefore $3 + (\frac{1}{13} + \frac{1}{19}) \approx 3.12$.

4. (20 points) Let $f(x) = e^x \sin x$.

(a) Compute $P_2(x)$, the second Taylor polynomial for $f(x)$, at $x = 0$.

(b) Approximate $\int_0^{0.2} e^x \sin x \, dx$ by computing $\int_0^{0.2} P_2(x) \, dx$.

(c) Use the error term $R_2(x)$ to give a good upper bound for the error in the approximation in part (b).

Answer: (a) We start by computing $f'(x) = e^x (\sin x + \cos x)$, and then $f''(x) = e^x (\sin x + \cos x) + e^x (\cos x - \sin x) = 2e^x \cos x$, and then finally $f^{(3)}(x) = 2e^x (\cos x - \sin x)$. Therefore, $f(0) = 0$, $f'(0) = 1$, and $f''(0) = 2$.

We then compute $P_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 = x + x^2$.

(b) We now approximate

$$\int_0^{0.2} e^x \sin x \, dx \approx \int_0^{0.2} (x + x^2) \, dx = \left[ \frac{x^2}{2} + \frac{x^3}{3} \right]_0^{0.2} \approx 0.022667.$$

(c) We know that $R_2(x) = \frac{f^{(3)}(ξ)}{6} x^3$, where $ξ$ is between 0 and $x$. In this case, $0 \leq x \leq 0.2$, so $0 \leq ξ \leq 0.2$. We know that $f^{(3)}(ξ) = 2e^ξ (\cos ξ - \sin ξ)$. The largest value of $\cos ξ - \sin ξ$ occurs when $ξ = 0$, and that value is 1. The largest value of $e^ξ$ occurs when $ξ = 0.2$, and that value is $1.221403$. So the largest value of $R_2(x)$ is $\frac{2}{6} \cdot 0.2^3 \approx 0.001629$.

The error incurred in making the approximation is $\int_0^{0.2} R_2(x) \, dx \leq \int_0^{0.2} 0.001629 \, dx = 0.000326$. That is an upper bound for the absolute value of the error.

In this case, we can compute the integral exactly, and find that $\int_0^{0.2} e^x \sin x \, dx \approx 0.0228$.

5. (20 points) Let $a$ be a fixed positive real number. Define the sequence $x_n$ by

$$x_0 = 1, \quad x_{n+1} = \frac{1}{2} \left( x_n + \frac{a}{x_n} \right), \quad n = 0, 1, 2, \ldots$$

Show that the sequence $\{x_n\}$ is the same sequence that arises by using Newton’s method on the function $f(x) = x^2 - a$ with $p_0 = 1$.

Answer: Newton’s method says that $p_{n+1} = p_n - f(p_n)/f'(p_n)$. In this case, that gives

$$p_{n+1} = p_n - \frac{p_n^2 - a}{2p_n} = \frac{2p_n^2 - (p_n^2 - a)}{2p_n} = \frac{p_n^2 + a}{2p_n} = \frac{1}{2} \left( \frac{p_n^2 + a}{p_n} \right) = \frac{1}{2} \left( p_n + \frac{a}{p_n} \right).$$