Mathematics 414
Exam 2
Answers

1. *(10 points)* Evaluate

$$\lim_{h\to 0}\left(\frac{3+h}{3-h}\right)^{1/h}.$$

Answer: Let $C = (\frac{3+h}{3-h})^{1/h}$. Rather than evaluate $\lim_{h\to 0} C$, we evaluate $\lim_{h\to 0}\log C$. We have

$$\lim_{h\to 0}\log C = \lim_{h\to 0}\frac{\log(3+h)-\log(3-h)}{h} = \lim_{h\to 0}\frac{\frac{1}{3+h}+\frac{1}{3-h}}{1}$$
$$= \lim_{h\to 0}\left(\frac{1}{3+h}+\frac{1}{3-h}\right) = \lim_{h\to 0}\frac{(3-h)+(3+h)}{(3-h)(3+h)} = \frac{2}{3}.$$

Because $\lim_{h\to 0}\log C = \frac{2}{3}$, we conclude that $\lim_{h\to 0} C = e^{2/3}$.

2. *(50 points)* Estimate

$$\int_{-1}^{1} e^{-\sin x}\,dx$$

(a) Using Gaussian quadrature with $n = 3$.
(b) Using the composite trapezoidal rule with $n = 5$.
(c) Give an upper bound for the error in your estimate in part (b).
(d) Estimate the integral using the composite Simpson's rule, first with $n = 2$ and then with $n = 4$.
(e) Use the methods of Chapter 4, Section 6, to estimate the error that occurred in part (d) when you used $n = 4$.

Answer: Write $f(x) = e^{-\sin x}$ for simplicity.

(a) Gaussian quadrature with $n = 3$ tells us to estimate the integral by $\frac{5}{9}f(0.7745966692) + \frac{8}{9}f(0) + \frac{5}{9}f(-0.7745966692)$, which in this case gives approximately 2.2830391143.

(b) We have $h = \frac{b-a}{n} = 0.4$, so we need to compute $0.2(f(-1.0) + 2f(-0.6) + 2f(-0.2) + 2f(0.2) + 2f(0.6) + f(1.0)) \approx 2.2969632458$.

(c) We first compute $f'(x) = -\cos x e^{-\sin x}$ and then $f''(x) = \cos^2 x e^{-\sin x} + \sin x e^{-\sin x}$. It's obviously very hard to give an exact upper bound, but we can surely take 1 as an upper bound for $\cos x$ and $\sin x$, and $e^{-\sin x} \le e$, because $-\sin x \le 1$. So it's safe to assert that $f''(x) \le 2e$ for $x \in [-1, 1]$. Now we apply the error estimate:

$$\frac{b-a}{12}h^2 f''(\mu) \le \frac{1}{6}(0.4)^2(2e) \approx 0.145.$$

(d) Simpson's rule with $n = 2$ gives $h = \frac{b-a}{n} = 1$, so we evaluate $\frac{1}{3}(f(-1.0) + 4f(0) + f(1.0)) \approx 2.2502842585$.

Simpson's rule with $n = 4$ gives $h = 0.5$, and then we evaluate $\frac{1}{6}(f(-1.0) + 4f(-0.5) + 2f(0.0) + 4f(0.5) + f(1.0)) \approx 2.2813323009$.

(e) The difference between these 2 values is approximately 0.0310480425, and we can be conservative and use this value as 10ϵ, meaning that the error in the second estimate of the integral is approximately 0.003.

3. *(10 points)* Suppose that you are given the values of $f(x_0)$, $f(x_0+h)$, and $f(x_0+3h)$. Use all three values to estimate $f'(x_0)$, and give an error bound for your estimate.

Answer: We write

$$f(x_0+h) = f(x_0) + hf'(x_0) + \frac{h^2}{2}f''(x_0) + \frac{h^3}{6}f^{(3)}(\xi_1)$$
$$f(x_0+3h) = f(x_0) + 3hf'(x_0) + \frac{9h^2}{2}f''(x_0) + \frac{27h^3}{6}f^{(3)}(\xi_2)$$

Multiply the first equation by 9 and subtract the second one, and we have

$$9f(x_0+h) - f(x_0+3h) = 8f(x_0) + 6hf'(x_0) + \frac{h^3}{6}(9f^{(3)}(\xi_1) - 27f^{(3)}(\xi_2))$$

So

$$f'(x_0) = \frac{1}{6h}(-8f(x_0) + 9f(x_0+h) - f(x_0+3h)) - \frac{h^3}{2}\left(\frac{9f^{(3)}(\xi_1) - 27f^{(3)}(\xi_2)}{18}\right).$$

Somewhat more work than we have done can show that this becomes

$$f'(x_0) = \frac{1}{6h}(-8f(x_0) + 9f(x_0+h) - f(x_0+3h)) - \frac{h^3}{2}f^{(3)}(\xi).$$

4. *(30 points)* (a) Determine constants a, b, c, and d that will produce a quadrature formula

$$\int_{-1}^{1} f(x)\,dx = af(-1) + bf(1) + cf'(-1) + df'(1)$$

which has degree of precision 3. In other words, your formula should give the exact answer for $\int_{-1}^{1} x^n\,dx$ for $n = 0, \ldots, 3$.

(b) Use your formula to estimate

$$\int_{-1}^{1} e^{-\sin x}\,dx.$$

Answer: We have

$$\begin{aligned}
\int_{-1}^{1} 1\,dx = 2 &= \quad a + b \\
\int_{-1}^{1} x\,dx = 0 &= -a + b + \ \ c + \ \ d \\
\int_{-1}^{1} x^2\,dx = \frac{2}{3} &= \quad a + b - 2c + 2d \\
\int_{-1}^{1} x^3\,dx = 0 &= -a + b + 3c + 3d
\end{aligned}$$

Subtract the third equation from the first, and we have $2c - 2d = \frac{4}{3}$, or $6c - 6d = 4$. Subtract the second equation from the fourth, and we have $0 = 3c + 3d$, or $6c + 6d = 0$. Add, yielding $12c = 4$, or $c = \frac{1}{3}$, and then $d = -\frac{1}{3}$.

Substitute these into the second equation, and we have $-a+b = 0$, or $a = b$, and then we have $a = b = 1$. Our formula is therefore

$$\int_{-1}^{1} f(x)\,dx \approx f(-1) + f(1) + \frac{1}{3}f'(-1) - \frac{1}{3}f'(1).$$

(b) We can now try applying this formula to $f(x) = e^{-\sin x}$ and $f'(x) = -\cos x e^{-\sin x}$, and we get

$$\int_{-1}^{1} e^{-\sin x}\,dx \approx e^{-\sin(-1)} + e^{-\sin 1} + \frac{1}{3}(-\cos(-1)e^{-\sin(-1)}) - \frac{1}{3}(-\cos(1)e^{-\sin 1}) \approx 2.4106962962$$

This is not all that far from the numbers produced in the first problem.