Mathematics 414
Final Exam
Wednesday, December 13, 2006, 12:30 PM

This is an open book examination. You may also use your calculators.

Do all of your work in the blue booklets. Please work carefully and neatly. Please give reasons for all your answers. You will not get any credit if you guess at an answer, whether your guess is wrong or right.

1. *(20 points)* Let $f(x) = x - e^{-x}$.
   (a) Which theorem allows you to conclude that the equation $f(x) = 0$ has a solution in the interval $[0, 1]$?
   (b) Perform three steps of bisection on this interval, to estimate a solution to that equation.
   (c) Perform three steps of Newton's method, with a starting value of $x = 0.5$, to estimate a solution to that equation.
   (d) Write the equation as $x = e^{-x}$, and perform three iterations of the function e^{-x} starting at $x = 0.5$, to estimate a solution to the equation.
   (e) Apply Aitken's Δ^2-method to your results in part (d) to get an improved estimate of a solution.

2. *(10 points)* Suppose that $f(-1) = 3$, $f(0) = 4$, and $f(2) = 5$. Find the Lagrange interpolating polynomial which interpolates these values, and use it to estimate $f'(0)$. Give a bound on the error of this estimate in terms of $f^{(k)}(x)$.

3. *(20 points)* Consider the differential equation

$$\frac{dy}{dt} = \cos y, \qquad y(0) = 1$$

Using a value of $h = 1$, estimate the value of $y(1)$ in the following ways:
   (a) Euler's Method.
   (b) Modified Euler's Method.
   (c) Third-order Taylor Method.
   (d) Midpoint Method
   (e) Fourth-order Runge-Kutta Method.

4. *(10 points)* Derive the Adams-Moulton Two-Step Implicit Method for solving first-order differential equations:

$$\begin{aligned} w_0 &= \alpha \\ w_1 &= \alpha_1 \\ w_{i+1} &= w_i + \frac{h}{12}\big(5f(t_{i+1}, w_{i+1}) + 8f(t_i, w_i) - f(t_{i-1}, w_{i-1})\big) \end{aligned}$$

5. *(5 points)* Suppose that

$$p_n = \sum_{k=0}^{n} a_k.$$

Show that

$$\hat{p}_n = p_n + \frac{a_{n+1}^2}{a_{n+1} - a_{n+2}},$$

where, as usual, $\hat{p}_n$ is the new sequence resulting from applying the Aitken's Δ^2-method.

6. *(10 points)* Suppose that $f(-1) = 3$, $f(0) = 4$, and $f(2) = 5$. Find the free cubic spline interpolation $S(x)$ through these points. What is the value of $S'(0)$?

7. *(20 points)* (a) Evaluate

$$\lim_{h \to 0} \left(\frac{1-h}{1+h} \right)^{1/h}.$$

(b) Let

$$N(h) = \left(\frac{1-h}{1+h} \right)^{1/h}.$$

Compute $N(0.5)$ and $N(0.1)$.

(c) Let L be the limit computed in part (a). Assume that $L = N(h) + K_1 h + K_2 h^2 + K_3 h^3 + K_4 h^4 + \cdots$. Use extrapolation and the values of $N(0.5)$ and $N(0.1)$ to compute an $O(h^2)$ approximation to L.

(d) We can see that $N(h) = N(-h)$. (You do not need to verify this.) Use that equation to show that $K_1 = K_3 = K_5 = \cdots = 0$, so

$$L = N(h) + K_2 h^2 + K_4 h^4 + K_6 h^6 + \cdots.$$

(e) Use the result of part (d) and an extrapolation to compute an $O(h^4)$ approximation to L.

8. *(20 points)* Estimate the integral $\int_0^1 e^{-x}\, dx$ by:

(a) integrating a degree-4 Maclaurin polynomial for e^{-x}.

(b) using the midpoint method with $h = 0.5$.

(c) using the trapezoidal rule with $h = 0.5$.

(d) using Simpson's rule with $h = 0.5$.

(e) rewriting the integral as the differential equation

$$\frac{dy}{dt} = e^{-t}, y(0) = 1$$

and estimating $y(1)$ by taking $h = 1.0$ and using the fourth-order Runge-Kutta method.