Mathematics 414
Final Exam
Answers

1. *(20 points)* Let $f(x) = x - e^{-x}$.
   (a) Which theorem allows you to conclude that the equation $f(x) = 0$ has a solution in the interval $[0, 1]$?
   (b) Perform three steps of bisection on this interval, to estimate a solution to that equation.
   (c) Perform three steps of Newton's method, with a starting value of $x = 0.5$, to estimate a solution to that equation.
   (d) Write the equation as $x = e^{-x}$, and perform three iterations of the function e^{-x} starting at $x = 0.5$, to estimate a solution to the equation.
   (e) Apply Aitken's Δ^2-method to your results in part (d) to get an improved estimate of a solution.

Answer: (a) We know that $f(0) = -1$, and $f(1) = 0.6321$, so the Intermediate Value Theorem says that between 0 and 1, $f(x)$ must take on the value 0.

(b) We compute that $f(0.5) = -0.1065$, and then $f(0.75) = 0.2776$, and then $f(0.625) = 0.0897$, so a solution lies between 0.5 and 0.625.

(c) We compute $f'(x) = 1 + e^{-x}$. Recalling that Newton's method is $x_{n+1} = x_n - f(x_n)/f'(x_n)$, and setting $x_0 = 0.5$, we have $x_1 = 0.5663$, $x_2 = 0.5671$, and $x_3 = 0.5671$ as well.

(d) Let $g(x) = e^{-x}$. We have $x_1 = g(0.5) = 0.6065$, and $x_2 = g(x_1) = 0.5452$, and $x_3 = g(x_2) = 0.5797$.

(e) We have $\hat{x}_1 = x_1 - (x_2 - x_1)^2/(x_3 - 2x_2 + x_1) = 0.5673$.

2. *(10 points)* Suppose that $f(-1) = 3$, $f(0) = 4$, and $f(2) = 5$. Find the Lagrange interpolating polynomial which interpolates these values, and use it to estimate $f'(0)$. Give a bound on the error of this estimate in terms of $f^{(k)}(x)$.

Answer: We have

$$\begin{aligned} L_2(x) &= \frac{(x-0)(x-2)}{(-1-0)(-1-2)}3 + \frac{(x+1)(x-2)}{(0+1)(0-2)}4 + \frac{(x+1)(x-0)}{(2+1)(2-0)}5 \\ &= x^2 - 2x - 2(x+1)(x-2) + \frac{5}{6}(x^2 + x) \\ &= x^2 - 2x - 2(x^2 - x - 2) + \frac{5}{6}(x^2 + x) \end{aligned}$$

Now, it is easy to compute $L_2'(0) = -2 + 2 + \frac{5}{6} = \frac{5}{6}$.

For the error, remember that $f(x) = L_2(x) + \frac{f^{(3)}(\xi)}{6}(x+1)x(x-2)$ for $x, \xi \in [-1, 2]$. Apply the product rule, and we have $f'(0) = L_2'(0) + \frac{f^{(3)}(\xi)}{6}(1)(-2) = L_2'(0) - \frac{f^{(3)}(\xi)}{3}$ for some $\xi \in [-1, 2]$.

3. *(20 points)* Consider the differential equation

$$\frac{dy}{dt} = \cos y, \qquad y(0) = 1$$

Using a value of $h = 1$, estimate the value of $y(1)$ in the following ways:
   (a) Euler's Method.
   (b) Modified Euler's Method.
   (c) Third-order Taylor Method.
   (d) Midpoint Method
   (e) Fourth-order Runge-Kutta Method.

Answer: (a) Euler's method has

$$\begin{aligned} &w_0 = 1 \\ &w_1 = w_0 + hf(t_0, w_0) = 1 + \cos 1 \approx 1.5403 \end{aligned}$$

(b) The Modified Euler's method has

$$\begin{aligned} w_0 &= 1 \\ w_1 &= w_0 + \frac{h}{2}\big(f(t_0, w_0) + f(t_1, w_0 + hf(t_0, w_0))\big) = 1 + \frac{1}{2}(\cos 1 + \cos(1 + \cos 1)) \approx 1.2854 \end{aligned}$$

(c) We must first compute $T^{(3)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2}f'(t_i, w_i) + \frac{h^2}{6}f''(t_i, w_i)$. We have $f(t_i, w_i) = \cos w_i$. Then $f'(t_i, w_i) = -\sin w_i \frac{dw_i}{dt} = -\sin w_i \cos w_i = -\frac{1}{2}\sin 2w_i$, and $f''(t_i, w_i) = -\cos 2w_i \frac{dw_i}{dt} = -\cos 2w_i \cos w_i$.

We therefore have

$$\begin{aligned} w_0 &= 1 \\ w_1 &= w_0 + hT^{(3)}(t_0, w_0 2) = 1 + \cos 1 - \frac{1}{4}\sin 2 - \frac{1}{6}\cos 2\cos 1 \approx 1.3505 \end{aligned}$$

(d) The Midpoint Method gives

$$\begin{aligned} w_0 &= 1 \\ w_1 &= w_0 + hf\left(t_i + \frac{h}{2}, w_i + \frac{h}{2}f(t_i, w_i)\right) = 1 + \cos(1 + \frac{\cos 1}{2}) = 1.2961 \end{aligned}$$

(e) Fourth-order Runge-Kutta gives

$$\begin{aligned} w_0 &= 1 \\ k_1 &= hf(t_0, w_0) = \cos 1 \approx 0.5403 \\ k_2 &= hf\left(t_0 + \frac{h}{2}, w_0 + \frac{1}{2}k_1\right) = \cos\left(1 + \frac{0.5403}{2}\right) \approx 0.2961 \\ k_3 &= hf\left(t_0 + \frac{h}{2}, w_0 + \frac{1}{2}k_2\right) = \cos\left(1 + \frac{0.2961}{2}\right) \approx 0.4102 \\ k_4 &= hf(t_0, w_0 + k_3) = \cos(1 + 0.4102) \approx 0.1599 \\ w_1 &= w_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \approx 1.3522 \end{aligned}$$

4. *(10 points)* Derive the Adams-Moulton Two-Step Implicit Method for solving first-order differential equations:

$$\begin{aligned} w_0 &= \alpha \\ w_1 &= \alpha_1 \\ w_{i+1} &= w_i + \frac{h}{12}\big(5f(t_{i+1}, w_{i+1}) + 8f(t_i, w_i) - f(t_{i-1}, w_{i-1})\big) \end{aligned}$$

Answer: We derive this formula by applying the Newton backward difference formula to interpolate a quadratic polynomial $P_2(t)$ through the three points $(t_k, f(t_k, w_k))$ for $k = t_{i-1}, t_i, t_{i+1}$, and then applying the approximation $w_{i+1} = w_i + \int_{t_i}^{t_{i+1}} P_2(t)\,dt$. We have

$$P_2(t) = \sum_{k=0}^{2}(-1)^k \binom{-s}{k} \nabla^k f(t_{i+1}, w_{i+1})$$

We make the change of variables $t = t_{i+1} + sh$, with $dt = h\,ds$, and we have

$$\begin{aligned} \int_{t_i}^{t_{i+1}} P_2(t)\,dt &= \int_{t_i}^{t_{i+1}} \sum_{k=0}^{2}(-1)^k \binom{-s}{k} \nabla^k f(t_{i+1}, w_{i+1})\,dt \\ &= \sum_{k=0}^{2}(-1)^k h \nabla^k f(t_{i+1}, w_{i+1}) \int_{-1}^{0} \binom{-s}{k}\,ds \end{aligned}$$

The evaluation of the integral yields

$$\int_{-1}^{0} \binom{-s}{0} ds = \int_{-1}^{0} 1 \, ds = 1$$
$$\int_{-1}^{0} \binom{-s}{1} ds = \int_{-1}^{0} -s \, ds = \frac{1}{2}$$
$$\int_{-1}^{0} \binom{-s}{2} ds = \int_{-1}^{0} \frac{s^2+s}{2} \, ds = -\frac{1}{12}$$

Therefore,

$$\begin{aligned} w_{i+1} &= w_i + h(f(t_{i+1}, w_{i+1}) - \frac{1}{2}\nabla f(t_{i+1}, w_{i+1}) - \frac{1}{12}\nabla^2 f(t_{i+1}, w_{i+1})) \\ &= w_i + h(f(t_{i+1}, w_{i+1}) - \frac{1}{2}(f(t_{i+1}, w_{i+1}) - f(t_i, w_i)) - \frac{1}{12}(f(t_{i+1}, w_{i+1}) - 2f(t_i, w_i) + f(t_{i-1}, w_{i-1}))) \\ &= w_i + \frac{h}{12}(5f(t_{i+1}, w_{i+1}) + 8f(t_i, w_i) - f(t_{i-1}, w_{i-1})) \end{aligned}$$

5. *(5 points)* Suppose that

$$p_n = \sum_{k=0}^{n} a_k.$$

Show that

$$\hat{p}_n = p_n + \frac{a_{n+1}^2}{a_{n+1} - a_{n+2}},$$

where, as usual, $\hat{p}_n$ is the new sequence resulting from applying the Aitken's Δ^2-method.
Answer: Notice that $p_{n+1} - p_n = a_{n+1}$ and $p_{n+2} - p_{n+1} = a_{n+2}$. Therefore,

$$\begin{aligned} \hat{p}_n &= p_n - \frac{(\Delta p_n)^2}{\Delta^2 p_n} = p_n - \frac{(p_{n+1} - p_n)^2}{(p_{n+2} - p_{n+1}) - (p_{n+1} - p_n)} \\ &= p_n - \frac{a_{n+1}^2}{a_{n+2} - a_{n+1}} = p_n + \frac{a_{n+1}^2}{a_{n+1} - a_{n+2}} \end{aligned}$$

6. *(10 points)* Suppose that $f(-1) = 3$, $f(0) = 4$, and $f(2) = 5$. Find the free cubic spline interpolation $S(x)$ through these points. What is the value of $S'(0)$?
Answer: Let

$$S(x) = \begin{cases} d_0(x+1)^3 + c_0(x+1)^2 + b_0(x+1) + a_0 & x \leq 0 \\ d_1(x-0)^3 + c_1(x-0)^2 + b_1(x-0) + a_1 & x \geq 0 \end{cases}$$

We have $h_0 = 1$ and $h_1 = 2$. We have $c_0 = 0$, $c_2 = 0$, $a_0 = 3$, $a_1 = 4$, and $a_2 = 5$. The matrix equation

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 6 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{3}{2} \\ 0 \end{bmatrix}$$

gives $c_1 = -\frac{1}{4}$. Then we can compute that $b_0 = \frac{13}{12}$, $b_1 = \frac{5}{6}$, $d_0 = -\frac{1}{12}$, and $d_1 = \frac{1}{24}$. Therefore, we have

$$S(x) = \begin{cases} \frac{-(x+1)^3}{12} + \frac{13(x+1)}{12} + 3 & x \leq 0 \\ \frac{x^3}{24} - \frac{x^2}{4} + \frac{5x}{6} + 4 & x \geq 0 \end{cases}$$

and we can then compute that $S'(0) = \frac{5}{6}$.

7. *(20 points)* (a) Evaluate

$$\lim_{h\to 0}\left(\frac{1-h}{1+h}\right)^{1/h}.$$

(b) Let

$$N(h) = \left(\frac{1-h}{1+h}\right)^{1/h}.$$

Compute $N(0.5)$ and $N(0.1)$.

(c) Let L be the limit computed in part (a). Assume that $L = N(h) + K_1h + K_2h^2 + K_3h^3 + K_4h^4 + \cdots$. Use extrapolation and the values of $N(0.5)$ and $N(0.1)$ to compute an $O(h^2)$ approximation to L.

(d) We can see that $N(h) = N(-h)$. (You do not need to verify this.) Use that equation to show that $K_1 = K_3 = K_5 = \cdots = 0$, so

$$L = N(h) + K_2h^2 + K_4h^4 + K_6h^6 + \cdots.$$

(e) Use the result of part (d) and an extrapolation to compute an $O(h^4)$ approximation to L.

Answer: (a) Let $C = (\frac{1-h}{1+h})^{1/h}$. Rather than evaluate $\lim_{h\to 0} C$, we evaluate $\lim_{h\to 0}\log C$. We have

$$\begin{aligned}\lim_{h\to 0}\log C &= \lim_{h\to 0}\frac{\log(1-h)-\log(1+h)}{h} = \lim_{h\to 0}\frac{-\frac{1}{1-h}-\frac{1}{1+h}}{1}\\ &= \lim_{h\to 0} -\left(\frac{1}{1-h}+\frac{1}{1+h}\right) = \lim_{h\to 0} -\frac{(1+h)+(1-h)}{(1-h)(1+h)} = -2.\end{aligned}$$

Because $\lim_{h\to 0}\log C = -2$, we conclude that $\lim_{h\to 0} C = e^{-2}$.

(b) We have $N(0.5) = 0.1111$ and $N(0.1) = 0.1344$.

(c) Suppose that $e^{-2} = N(h) + K_1h + K_2h^2 + \cdots$. That gives

$$\begin{aligned}e^{-2} &= N(0.5) + 0.5K_1 + \cdots\\ e^{-2} &= N(0.1) + 0.1K_1 + \cdots\end{aligned}$$

Multiplying the second equation by 5 and subtracting the first one gives $4e^{-2} \approx 5N(0.1) - N(0.5)$, or $e^{-2} \approx \frac{5N(0.1)-N(0.5)}{4} \approx 0.1403$.

(d) We have

$$\begin{aligned}e^{-2} &= N(\ \ h) + K_1h + K_2h^2 + K_3h^3 + K_4h^4 + \cdots\\ e^{-2} &= N(-h) - K_1h + K_2h^2 - K_3h^3 + K_4h^4 + \cdots\end{aligned}$$

Adding and dividing by 2 yields

$$e^{-2} = N(h) + K_2h^2 + K_4h^4 + \cdots.$$

(e) Now we have

$$\begin{aligned}e^{-2} &= N(0.5) + 0.25K_2 + \cdots\\ e^{-2} &= N(0.1) + 0.01K_2 + \cdots\end{aligned}$$

Multiplying the second equation by 25 and subtracting the first one yields $24e^{-2} \approx 25N(0.1) - N(0.5)$, or $e^{-2} \approx \frac{25N(0.1)-N(0.5)}{24} \approx 0.1354$. The correct answer to 4 decimal places is 0.1353.

8. *(20 points)* Estimate the integral $\displaystyle\int_0^1 e^{-x}\,dx$ by:

(a) integrating a degree-4 Maclaurin polynomial for e^{-x}.

(b) using the midpoint method with $h = 0.5$.

(c) using the trapezoidal rule with $h = 0.5$.

(d) using Simpson's rule with $h = 0.5$.
(e) rewriting the integral as the differential equation

$$\frac{dy}{dt} = e^{-t}, y(0) = 1$$

and estimating $y(1)$ by taking $h = 1.0$ and using the fourth-order Runge-Kutta method.

Answer: (a) We have

$$e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \cdots$$

so we can approximate

$$\int_0^1 e^{-x}\,dx \approx \int_0^1 \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24}\right) dx = 1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} + \frac{1}{120} \approx 0.6333$$

(b) The midpoint method yields

$$\int_0^1 e^{-x}\,dx \approx \frac{1}{2}(e^{-0.25} + e^{-0.75}) \approx 0.6256.$$

(c) The trapezoidal rule yields

$$\int_0^1 e^{-x}\,dx \approx \frac{1}{4}(e^{0} + 2e^{-0.5} + e^{-1}) \approx 0.6452.$$

(d) Simpson's rule yields

$$\int_0^1 e^{-x}\,dx \approx \frac{1}{6}(e^{0} + 4e^{-0.5} + e^{-1}) \approx 0.6323.$$

(e) We have

$$\begin{aligned}
w_0 &= 0\\
k_1 &= e^{-0} = 1\\
k_2 &= e^{-0.5} \approx 0.6065\\
k_3 &= e^{-0.5} \approx 0.6065\\
k_4 &= e^{-1.0} \approx 0.3679\\
w_1 &= 0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \approx 0.6323
\end{aligned}$$

which not coincidentally is the same answer as Simpson's method.