MT414: Numerical Analysis
Homework 2
Answers

1. Let $a = 0.96$ and $b = 0.99$.

 (a) Using two-digit rounding arithmetic, compute $\dfrac{a+b}{2}$.

 (b) Using two-digit rounding arithmetic, compute $a + \dfrac{b-a}{2}$.

 (c) Which of these two values is a better approximation to the actual value of $\dfrac{a+b}{2}$?

Answer: (a) We compute that $a+b$ rounds to 2.0 using two-digit rounding arithmetic, and therefore $\dfrac{a+b}{2}$ rounds to 1.0.

(b) Now, we compute that $\dfrac{b-a}{2}$ rounds to 0.015, and $a + 0.15$ rounds to 0.98.

(c) The result in (b) is considerably better than that in (a), because it is between the values of a and b. The other result could lead to serious errors.

2. Find the rates of convergence of the following functions as $n \to \infty$:

a. $\displaystyle\lim_{n\to\infty} \sin\frac{1}{n} = 0$

b. $\displaystyle\lim_{n\to\infty} \sin\frac{1}{n^2} = 0$

c. $\displaystyle\lim_{n\to\infty} \left(\sin\frac{1}{n}\right)^2 = 0$

d. $\displaystyle\lim_{n\to\infty} \log(n+1) - \log(n) = 0$

Answer: The first three problems can be answered much more easily if we know that $\sin x \le x$ for $0 \le x < 1$. (Much more than this is true, but this inequality suffices.) As a result, we have $|\sin\left(\frac{1}{n}\right)| < \frac{1}{n}$, and so $\sin\frac{1}{n} = O(\frac{1}{n})$. Similarly, $|\sin\left(\frac{1}{n^2}\right)| < \left|\frac{1}{n^2}\right|$, and so $\sin\left(\frac{1}{n^2}\right) = O(\frac{1}{n^2})$. We also can take the inequality $|\sin\left(\frac{1}{n}\right)| < \frac{1}{n}$ and square both sides, giving $|\sin\left(\frac{1}{n}\right)|^2 < \frac{1}{n^2}$, and therefore $\left(\sin\frac{1}{n}\right)^2 = O(\frac{1}{n^2})$.

The last one is a bit more interesting. We rewrite $\log(n+1)-\log(n)$ as $\log\left(1+\frac{1}{n}\right)$, and now use the fact that $|\log(1+x)| < |x|$ for $0 < x < 1$. Therefore, $|\log(n+1)-\log(n)| < \frac{1}{n}$, and so $\log(n+1) - \log(n) = O(\frac{1}{n})$.

3. Find the rates of convergence of the following functions as $h \to 0$:

a. $\displaystyle\lim_{h\to 0} \frac{\sin h}{h} = 1$

b. $\displaystyle\lim_{h\to 0} \frac{1-\cos h}{h} = 0$

c. $\displaystyle\lim_{h\to 0} \frac{\sin h - h\cos h}{h} = 0$

d. $\displaystyle\lim_{h\to 0} \frac{1-e^h}{h} = -1$

Answer: Here, Maclaurin series are the easiest way to get a solution:

$$\frac{\sin h}{h} = \frac{h - \frac{h^3}{6} + \cdots}{h} = 1 - \frac{h^2}{6} + \cdots$$

and so $\frac{\sin h}{h} = 1 + O(h^2)$.

For **b**, we have

$$\frac{1-\cos h}{h} = \frac{1-(1-\frac{h^2}{2}+\cdots)}{h} = \frac{h}{2}+\cdots,$$

so $\frac{1-\cos h}{h} = O(h)$.

For **c**, we have

$$\frac{\sin h - h\cos h}{h} = \frac{(h-\frac{h^3}{6}+\cdots)-h(1-\frac{h^2}{4}+\cdots)}{h} = \frac{-h^2}{6}+\frac{h^2}{4}$$

so $\frac{\sin h - h\cos h}{h} = O(h^2)$.

Finally, for **d**, we have

$$\frac{1-e^h}{h} = \frac{1-(1+h+\frac{h^2}{2}+\cdots)}{h} = -1-\frac{h}{2}+\cdots,$$

so $\frac{1-e^h}{h} = -1+O(h)$.

4. Suppose that $0 < q < p$ and $\alpha_n = \alpha + O(n^{-p})$. Show that $\alpha_n = \alpha + O(n^{-q})$.

Answer: The definition says that for sufficiently large n and for some positive constant K, $|\alpha_n - \alpha| < Kn^{-p}$. Because $q < p$, we know that $n^{-p} < n^{-q}$. Therefore, $|\alpha_n - \alpha| < Kn^{-q}$, which in turn says that $\alpha_n = \alpha + O(n^{-q})$.

5. Suppose that $0 < q < p$ and $F(h) = L + O(h^p)$. Show that $F(h) = L + O(h^q)$.

Answer: The definition says that for sufficiently small positive real numbers h and some positive constant K, $|F(h) - L| < K|h^p|$. Again, because $q < p$ and $|h| < 1$, $|h^p| < |h^q|$. This means that $|F(h) - L| < K|h^q|$, which in turn means that $F(h) = L + O(h^q)$.

6. Use the bisection method to find a solution accurate to within 0.01 for the equation $x^4 - 2x^3 - 4x^2 + 4x + 4 = 0$ on the interval $[-1, 4]$.

Answer: Here is a chart of the results, with a the left-hand endpoint of the bounding interval, b the right-hand endpoint of the bounding interval, and m the midpoint of the bounding interval at each stage:

n	a	b	m	$f(m)$
1	-1	4	1.5000	-0.6875
2	1.5000	4	2.7500	0.3477
3	1.5000	2.7500	2.1250	-4.3630
4	2.1250	2.7500	2.4375	-3.6797
5	2.4375	2.7500	2.5938	-2.1745
6	2.5938	2.7500	2.6719	-1.0526
7	2.6719	2.7500	2.7109	-0.3888
8	2.7109	2.7500	2.7305	-0.0299
9	2.7305	2.7500	2.7402	0.1565
10	2.7305	2.7402	2.7354	0.0627

This tells us that a root is between 2.7305 and 2.7354.

7. Let $f(x) = x^4 + 2x^2 - x - 3$. Use algebraic manipulations to show that each of the following functions has a fixed point at p if and only if $f(p) = 0$:

a. $g_1(x) = (3 + x - 2x^2)^{1/4}$

b. $g_2(x) = \left(\frac{x + 3 - x^4}{2}\right)^{1/2}$

c. $g_3(x) = \left(\frac{x + 3}{x^2 + 2}\right)^{1/2}$

d. $g_4(x) = \frac{3x^4 + 2x^2 + 3}{4x^3 + 4x - 1}$

Answer: (a) Start with $x^4 + 2x^2 - x - 3 = 0$, and move the last three terms to the right-hand side of the equation, yielding $x^4 = -2x^2 + x + 3$. Take fourth roots, and we have $x = (3 + x - 2x^2)^{1/4}$. So a fixed point of $g_1(x) = (3 + x - 2x^2)^{1/4}$ will be a root of the original equation. The algebra here is reversible, yielding the "if and only if" conclusion.

(b) Start with $x^4 + 2x^2 - x - 3 = 0$, and now move all but the quadratic term to the right-hand side of the equation, yielding $2x^2 = -x^4 + x + 3$. Divide by 2 and take square roots to get $x = ((3 + x - x^4)/2)^{1/2}$. Again, we can see that this process is reversible.

(c) Start with $x^4 + 2x^2 - x - 3 = 0$, and move the last two terms to the right-hand side, yielding $x^4 + 2x^2 = x + 3$. Factor the left-hand side into $x^2(x^2 + 2)$, divide by $x^2 + 2$, and take square roots, and we get $x = ((x + 3)/(x^2 + 2))^{1/2}$.

(d) This is Newton's method in disguise. Start with $x^4 + 2x^2 - x - 3 = 0$, and divide both sides by $-(4x^3 + 4x - 1)$, yielding $-(x^4 + 2x^2 - x - 3)/(4x^3 + 4x - 1) = 0$. Now add x to both sides, yielding $x - (x^4 + 2x^2 - x - 3)/(4x^3 + 4x - 1) = x$. Simplify the left-hand side, and we get $(3x^4 + 2x^2 + 3)/(4x^3 + 4x - 1) = x$. Again, the process is reversible.

8. Use the functions $g_k(x)$ in the previous problem and perform four iterations (if possible, without dividing by 0 or taking the square root of a negative number), starting with $p_0 = 1$ and $g_k(p_n) = p_{n+1}$. Which of the four functions seems to give the best approximation to a solution of the equation $f(x) = 0$?

Answer: I computed the following:

n	$g_1(p_n)$	$g_2(p_n)$	$g_3(p_n)$	$g_4(p_n)$
0	1.1892	1.2247	1.1547	1.1429
1	1.0801	0.9937	1.1164	1.1245
2	1.1497	1.2286	1.1261	1.1241
3	1.1078	0.9875	1.1236	1.1241
4	1.1339	1.2322	1.1242	1.1241

Clearly, $g_4(x)$ is converging to the fixed point quicker than any of the other three functions.