MT414: Numerical Analysis
Homework 3
Answers

1. On last week's homework, we used the bisection method to find a solution for the equation $x^4 - 2x^3 - 4x^2 + 4x + 4 = 0$ on the interval $[-1, 4]$.

(a) Perform 4 iterations of Newton's method to solve the same equation with $p_0 = -1$.

(b) Perform 4 iterations of Newton's method to solve the same equation with $p_0 = 4$.

(c) Perform 4 iterations of the secant method with $p_0 = -1$ and $p_1 = 4$ to solve the same equation.

(d) Perform 4 iterations of the secant method with $p_0 = 4$ and $p_1 = -1$ to solve the same equation.

(e) Perform 4 iterations of the method of false position with $p_0 = -1$ and $p_1 = 4$ to solve the same equation.

Answer: (a) We set $g(x) = x - (x^4 - 2x^3 - 4x^2 + 4x + 4)/(4x^3 - 6x^2 - 8x + 4)$, $p_0 = -1$, and $g(p_{n-1}) = p_n$. I compute $p_1 = -0.5000$, $p_2 = -0.7188$, $1p_3 = -0.7319$, and $p_4 = -0.7321$.

(b) With the same definition of $g(x)$ and $p_0 = 4$, I compute $p_1 = 3.3636$, $p_2 = 2.9714$, $p_3 = 2.7834$, and $p_4 = 2.7352$.

(c) Now we set $p_{n+1} = g(p_n) = p_n - f(p_n)(p_n - p_{n-1})/(f(p_n) - f(p_{n-1}))$, where $f(x) = x^4 - 2x^3 - 4x^2 + 4x + 4$. We start with $p_0 = -1$ and $p_1 = 4$, and I compute $p_2 = -0.9412$, $p_3 = -0.8913$, $p_4 = -0.6748$, and $p_5 = -0.7402$.

(d) We do the same thing, with $p_0 = 4$ and $p_1 = -1$, and now I compute $p_2 = -0.9412$, $p_3 = -0.5918$, $p_4 = -0.7572$, and $p_5 = -0.7340$.

(e) This one is a bit trickier. We use the essentially the same formula as above, starting with $p_0 = -1$ and $p_1 = 4$, to get $p_2 = -0.9412$. Because $f(-0.9412) < 0$, we now do the same thing with the two points -0.9412 and 4, getting $p_3 = -0.8913$. Because $f(-0.8913) < 0$, we now do the same thing with the two points -0.8913 and 4, getting $p_4 = -0.8512$. Because $f(-0.8512) < 0$, we now use the two points -0.8512 and 4 to get $p_5 = -0.8199$.

2. Let $f(x) = x \sin x$.

(a) Show that $f(x)$ has a double zero at $x = 0$.

(b) Let $p_0 = 1.5$, and perform 3 iterations of Newton's method to try to find the root.

(c) Let $\mu(x) = f(x)/f'(x)$. Perform 3 iterations of Newton's method using the function $\mu(x)$ to try to find the root. Is the convergence noticeably quicker than for $f(x)$?

Answer: (a) We have $f'(x) = x \cos x + \sin x$, and so $f(0) = f'(0) = 0$. On the other hand, $f''(x) = -x \sin x + 2 \cos x$, and so $f''(0) = 2$.

(b) We iterate $g(x) = x - x \sin x/(x \cos x + \sin x)$, and get $p_1 = 0.1442$, $p_2 = 0.0719$, and $p_3 = 0.0359$.

(c) Setting $\mu(x) = x \sin x/(\sin x + x \cos x)$, and iterating $g_1(x) = x - \mu(x)/\mu'(x)$, I compute $p_1 = 0.9911$, $p_2 = 0.3112$, and $p_3 = 0.0100$. Though the initial point is further from the solution, obviously we have converged quicker by p_3.

3. The *ordinary annuity equation* is

$$A = \frac{P}{i}(1 - (1+i)^{-n}),$$

where A is the amount of money to be borrowed, P is the amount of each payment, i is the interest rate per period, and there are n equally spaced payments. Suppose that a buyer needs a 30-year home mortgage of \$135,000, with payments of at most \$1,000 per month. (This means that there are 360 payments in all.) What is the maximal annual interest rate that the buyer can afford?

Answer: We have $A = 135000$, $P = 1000$, and $n = 360$, and we need to solve for i. In other words, we have the equation $135000i = 1000(1 - (1+i)^{-360})$. Divide by 1000, and let $f(x) = 135x - 1 + (1+x)^{-360}$, and iterate $g(x) = x - f(x)/f'(x)$ with $p_0 = 0.5$, to get $p_1 = 0.0074$, and $p_2 = 0.0068$ and $p_3 = 0.0067$. The solution appears to be stable at that point.

Thus, the monthly interest rate is .67%. To find the annual interest rate, add 1 and raise to the twelfth power (because there are 12 months in a year), yielding 1.0841. This works out to an annual percentage rate of 8.41%. If instead, you multiply the monthly rate by 12 (which is not really correct), you get an annual rate of 8.1%.

4. Suppose that $f(x)$ has m continuous derivatives (in our usual notation, f is C^m). Modify the proof of Theorem 2.10 in the text to show that f has a zero of multiplicity m at p if and only if $f(p) = f'(p) = f''(p) = \cdots = f^{(m-1)}(p) = 0$ and $f^{(m)}(p) \neq 0$.

Answer: Assume first that f has a root of multiplicity m. Then we can write $f(x) = (x-p)^m q(x)$, where $q(p) \neq 0$. Then $x - p$ will divide $f'(x)$ and $f''(x)$ all the way up to $f^{(m-1)}(x)$, meaning that $f'(p) = f''(p) = f^{(3)}(p) = \cdots = f^{(m-1)}(p) = 0$. However, $f^{(m)}(p) = m!q(p) \neq 0$.

On the other hand, assume that $f(p) = f'(p) = f''(p) = \cdots = f^{(m-1)}(p) = 0$ and $f^{(m)}(p) \neq 0$. We can write $f(x)$ as a degree m Taylor polynomial:

$$f(x) = f(p) + (x-p)f'(p) + \frac{(x-p)^2}{2!}f''(p) + \cdots + \frac{f^{(m-1)}(p)}{(m-1)!}(x-p)^{m-1} + \frac{f^{(m)}(\xi)}{m!}(x-p)^m.$$

Most of the terms vanish, and we are left with $f(x) = (x-p)^m q(x)$, where $q(x)$ is defined to be the necessary factor to make this equation work.

5. Given a function $f(x)$ with continuous second derivative, let

$$g(x) = x - \frac{f(x)}{f'(x)} - \frac{f''(x)}{2f'(x)}\left(\frac{f(x)}{f'(x)}\right)^2.$$

(*a*) Suppose that $f(p) = 0$. Show that $g'(p) = g''(p) = 0$. This means (you do not need to check this) that often the series $p_n = g(p_{n-1})$ will converge cubically.

(*b*) Let $f(x) = x^4 - 2x^3 - 4x^2 + 4x + 4$. Iterate $g(x)$ twice with a starting point of $p_0 = -1$. Is the result better than using the standard Newton's method?

Answer: We compute

$$
\begin{aligned}
g'(x) &= 1 - \frac{f'(x)^2 - f(x)f''(x)}{f'(x)^2} - \left(\frac{2f'(x)f^{(3)}(x) - 2f''(x)^2}{4f'(x)^2}\right)\left(\frac{f(x)}{f'(x)}\right)^2 \\
&\quad - \frac{f''(x)}{f'(x)}\left(\frac{f(x)}{f'(x)}\right)\left(\frac{f'(x)^2 - f(x)f''(x)}{f'(x)^2}\right) \\
&= \frac{3f(x)^2 f''(x)^2}{2f'(x)^4} - \frac{f(x)^2 f^{(3)}(x)}{2f'(x)^3}
\end{aligned}
$$

We can now use the fact that $f(p) = 0$ to conclude that $g'(p) = 0$.

We can do more, by noticing that we can write $g'(x) = f(x)^2 q(x)$, and therefore $g''(x) = f(x)^2 q'(x) + 2f(x)q(x)$, which lets us see instantly that $g''(p) = 0$.

(b) I compute that $p_1 = -1.5000$ and $p_2 = -1.4170$. A further iteration yields $p_3 = -1.4142$, and the sequence seems to stabilize at this point.