

MT414: Numerical Analysis
Homework 4
Answers

1. Let $p_n = \frac{1}{n}$. Use the Aitken's Δ^2 -method to compute \hat{p}_n .

Answer: We have

$$\begin{aligned}\hat{p}_n &= p_n - \frac{(p_{n+1} - p_n)^2}{p_{n+2} - 2p_{n+1} + p_n} = \frac{1}{n} - \frac{\left(\frac{1}{n+1} - \frac{1}{n}\right)^2}{\frac{1}{n+2} - \frac{2}{n+1} + \frac{1}{n}} \\ &= \frac{1}{n} - \frac{\left(\frac{1}{n(n+1)}\right)^2}{\left(\frac{2}{n(n+1)(n+2)}\right)} = \frac{1}{n} - \frac{n+2}{2n(n+1)} = \frac{1}{2(n+1)}.\end{aligned}$$

Notice that this is not a large improvement in the rate of convergence.

2. A sequence $\{p_n\}$ is *superlinearly convergent* to the limit p if

$$\lim_{n \rightarrow \infty} \left| \frac{p_{n+1} - p}{p_n - p} \right| = 0.$$

(a) Show that if $p_n \rightarrow p$ of order $\alpha > 1$, then $\{p_n\}$ is superlinearly convergent to p .

(b) Let $p_n = \frac{1}{n^n}$. Show that p_n is superlinearly convergent to 0, but that p_n does not converge to 0 with any order $\alpha > 1$.

Answer: (a) Suppose that

$$\lim_{n \rightarrow \infty} \frac{|p_{n+1} - p_n|}{|p_n - p|^\alpha} = \lambda,$$

where $\alpha > 1$ and $\lambda \neq 0$. Then

$$\lim_{n \rightarrow \infty} \left| \frac{p_{n+1} - p}{p_n - p} \right| = \lim_{n \rightarrow \infty} \frac{|p_{n+1} - p_n|}{|p_n - p|^\alpha} |p_n - p|^{\alpha-1} = \lambda \lim_{n \rightarrow \infty} |p_n - p|^{\alpha-1} = \lambda \cdot 0 = 0.$$

(b) First, we check that the convergence is superlinear:

$$\lim_{n \rightarrow \infty} \frac{1}{\frac{(n+1)^{n+1}}{n^n}} = \lim_{n \rightarrow \infty} \frac{n^n}{(n+1)^{n+1}} = \lim_{n \rightarrow \infty} \left(\frac{n}{n+1}\right)^n \left(\frac{1}{n+1}\right) = e^{-1} \cdot 0 = 0,$$

where we have used the fact that $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$ to conclude that $\lim_{n \rightarrow \infty} \left(\frac{n}{n+1}\right)^n = \frac{1}{e}$.

On the other hand, suppose that $\alpha > 1$. We compute

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{1}{\left(\frac{1}{n^n}\right)^\alpha} &= \lim_{n \rightarrow \infty} \frac{n^{\alpha n}}{(n+1)^{n+1}} = \left(\lim_{n \rightarrow \infty} \frac{n^n}{(n+1)^n}\right) \left(\lim_{n \rightarrow \infty} \frac{n^{(\alpha-1)n}}{n+1}\right) = e^{-1} \left(\lim_{n \rightarrow \infty} \frac{n^{(\alpha-1)n}}{n+1}\right) \\ &= e^{-1} \left(\lim_{n \rightarrow \infty} \frac{n}{n+1}\right) \lim_{n \rightarrow \infty} n^{(\alpha-1)n-1} = e^{-1} \lim_{n \rightarrow \infty} n^{(\alpha-1)n-1} = \infty\end{aligned}$$

3. Suppose that we have the following values for a function $f(x)$:

x	$f(x)$
2.1	1.5602
2.2	1.4905
2.4	1.3833
2.5	1.3415

- (a) Compute 2 different quadratic Lagrange interpolating polynomials using first the points 2.1, 2.2, and 2.4, and then using the points 2.2, 2.4, and 2.5.
 (b) Compute the cubic Lagrange interpolating polynomial passing through all 4 of these points.
 (c) Using each of those 3 polynomials, estimate the value of $f(2.3)$.

Answer: (a) We compute

$$\frac{(x-2.2)(x-2.4)}{(2.1-2.2)(2.1-2.4)}1.5602 + \frac{(x-2.1)(x-2.4)}{(2.2-2.1)(2.2-2.4)}1.4905 + \frac{(x-2.1)(x-2.2)}{(2.4-2.1)(2.4-2.2)}1.3833$$

$$= 52.0067(x-2.2)(x-2.4) - 74.5250(x-2.1)(x-2.4) + 23.0550(x-2.1)(x-2.2) = 0.5367x^2 - 3.0047x + 5.5033$$

We also compute

$$\frac{(x-2.4)(x-2.5)}{(2.2-2.4)(2.2-2.5)}1.4905 + \frac{(x-2.2)(x-2.5)}{(2.4-2.2)(2.4-2.5)}1.3833 + \frac{(x-2.2)(x-2.4)}{(2.5-2.2)(2.5-2.4)}1.3415$$

$$= 24.8417(x-2.4)(x-2.5) - 69.1650(x-2.2)(x-2.5) + 44.7167(x-2.2)(x-2.4) = 0.3933x^2 - 2.3453x + 4.7465$$

(b) The cubic polynomial is

$$\frac{(x-2.2)(x-2.4)(x-2.5)}{(2.1-2.2)(2.1-2.4)(2.1-2.5)}1.5602 + \frac{(x-2.1)(x-2.4)(x-2.5)}{(2.2-2.1)(2.2-2.4)(2.2-2.5)}1.4905$$

$$+ \frac{(x-2.1)(x-2.2)(x-2.5)}{(2.4-2.1)(2.4-2.2)(2.4-2.5)}1.3833 + \frac{(x-2.1)(x-2.2)(x-2.4)}{(2.5-2.1)(2.5-2.2)(2.5-2.4)}1.3415$$

$$= -130.0167(x-2.2)(x-2.4)(x-2.5) + 248.4167(x-2.1)(x-2.4)(x-2.5)$$

$$- 230.5500(x-2.2)(x-2.5)(x-2.1) + 111.7917(x-2.1)(x-2.2)(x-2.4)$$

$$= -0.3583x^3 + 2.9375x^2 - 8.3582x + 9.4765$$

(c) The first polynomial gives 1.4316, the second gives 1.4329, and the third gives 1.4322.

4. Suppose that we have the following values for a function $g(x)$:

x	$f(x)$
3.3	2.6834
3.4	2.9812
3.5	3.3234
3.7	4.1707

Use Neville's method to estimate $g(3.6)$.

Answer: We have

3.3	2.6834			
		3.5768		
3.4	2.9812		3.7100	
		3.6656		3.7174
3.5	3.3234		3.7199	
		3.7471		
3.7	4.1707			