MT414: Numerical Analysis
Homework 5
Answers

1. Suppose that we have the following values for a function $f(x)$:

x	$f(x)$
2.1	1.5602
2.2	1.4905
2.4	1.3833
2.5	1.3415

Compute the free cubic spline interpolation for $f(x)$, and use it to estimate the value of $f(2.3)$.

Answer: We have $a_0 = 1.5602$, $a_1 = 1.4905$, $a_2 = 1.3833$, and $a_3 = 1.3415$. We also have $h_0 = 0.1$, $h_1 = 0.2$, and $h_2 = 0.1$. As a result, our matrix equation becomes:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.1 & 0.6 & 0.2 & 0 \\ 0 & 0.2 & 0.6 & 0.1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.4830 \\ 0.3540 \\ 0 \end{bmatrix}$$

We can ignore first and last rows and first and last columns, because $c_0 = c_3 = 0$, and get the augmented matrix:

$$\begin{bmatrix} 0.6 & 0.2 & \mid & 0.4830 \\ 0.2 & 0.6 & \mid & 0.3540 \end{bmatrix}$$

Multiply the second row by 3, and we get

$$\begin{bmatrix} 0.6 & 0.2 & \mid & 0.4830 \\ 0.6 & 1.8 & \mid & 1.0620 \end{bmatrix}$$

Now add -1 times the first row to the second, and we have:

$$\begin{bmatrix} 0.6 & 0.2 & \mid & 0.4830 \\ 0.0 & 1.6 & \mid & 0.5790 \end{bmatrix}$$

So $c_2 = 0.5790/1.6 = 0.3619$, and then $c_1 = 0.6844$.

Now, we apply $d_j = \dfrac{c_{j+1} - c_j}{3h_j}$ to conclude that $d_2 = -1.2063$, $d_1 = -0.5375$, and $d_0 = 2.2812$. Applying $b_j = \dfrac{a_{j+1} - a_j}{3} - \dfrac{h_j}{3}(2c_j + c_{j+1})$ yields $b_0 = -0.7198$, $b_1 = -0.6514$, and $b_2 = -0.4421$.

So our answer becomes

$$S(x) = \begin{cases} 1.5602 - 0.7198(x-2.1) + 2.2812(x-2.1)^3 & 2.1 \le x \le 2.2 \\ 1.4905 - 0.6514(x-2.2) + 0.6844(x-2.2)^2 - 0.5375(x-2.2)^3 & 2.2 \le x \le 2.4 \\ 1.3833 - 0.4421(x-2.4) + 0.3619(x-2.4)^2 - 1.2063(x-2.4)^3 & 2.4 \le x \le 2.5 \end{cases}$$

Though it is not necessary, you can simplify this to:

$$S(x) = \begin{cases} +2.2812x^3 - 14.3716x^2 + 29.4604x - 18.0544 & 2.1 \le x \le 2.2 \\ -0.5375x^3 + \ \ 4.2319x^2 - 11.4672x + 11.9594 & 2.2 \le x \le 2.4 \\ -1.2063x^3 + \ \ 9.0473x^2 - 23.0241x + 21.2048 & 2.4 \le x \le 2.5 \end{cases}$$

We can estimate $f(2.3)$ by computing $S(2.3) = 1.4316$.

2. Suppose that we have the following values for a function $g(x)$:

x	$f(x)$
3.3	2.6834
3.4	2.9812
3.5	3.3234
3.7	4.1707

Compute the free cubic spline interpolation for $g(x)$, and use it to estimate the value of $g(3.6)$.
Answer: We have $(a_0, a_1, a_2, a_3) = (2.6834, 2.9812, 3.3234, 4.1707)$. We also have $(h_0, h_1, h_2) = (0.1, 0.1, 0.2)$. As a result, our matrix equation becomes:

$$\begin{bmatrix} 1.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.1000 & 0.4000 & 0.1000 & 0.0000 \\ 0.0000 & 0.1000 & 0.6000 & 0.2000 \\ 0.0000 & 0.0000 & 0.0000 & 1.0000 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0.0000 \\ 1.3320 \\ 2.4435 \\ 0.0000 \end{bmatrix}$$

We can again ignore first and last rows and first and last columns, because $c_0 = c_3 = 0$, and get the augmented matrix:

$$\begin{bmatrix} 0.4000 & 0.1000 & | & 1.3320 \\ 0.1000 & 0.6000 & | & 2.4435 \end{bmatrix}$$

Divide the first row by 4, and we have

$$\begin{bmatrix} 0.1000 & 0.0250 & | & 0.3330 \\ 0.1000 & 0.6000 & | & 2.4435 \end{bmatrix}$$

Multiply by -1 and add to the second, and we get

$$\begin{bmatrix} 0.1000 & 0.0250 & | & 0.3330 \\ 0.0000 & 0.5750 & | & 2.1105 \end{bmatrix}$$

So $c_2 = 3.6704$, and then $c_1 = 2.4124$. We then compute $(b_0, b_1, b_2) = (2.8976, 3.1388, 3.7471)$ and $(d_0, d_1, d_2) = (8.0413, 4.1935, -6.1174)$.

So our answer becomes

$$S(x) = \begin{cases} 2.2834 + 2.8976(x - 3.3) + 8.0413(x - 3.3)^3 & 3.3 \le x \le 3.4 \\ 2.9812 + 3.1388(x - 3.4) + 2.4124(x - 3.4)^2 + 4.1935(x - 3.4)^3 & 3.4 \le x \le 3.5 \\ 3.3234 + 3.7471(x - 3.5) + 3.6704(x - 3.5)^2 - 6.1174(x - 3.5)^3 & 3.5 \le x \le 3.7 \end{cases}$$

Again, we can simplify this to:

$$S(x) = \begin{cases} +8.0413x^3 - 79.6089x^2 + 265.6069x - 296.2589 & 3.3 \le x \le 3.4 \\ +4.1935x^3 - 40.3613x^2 + 132.1651x - 144.6247 & 3.4 \le x \le 3.5 \\ -6.1174x^3 + 67.9031x^2 - 246.7602x + 297.4545 & 3.5 \le x \le 3.7 \end{cases}$$

We have $S(3.6) = 3.7287$.

3. Consider the following table of values of the sine function:

x	$\sin(x)$
0	0
$\frac{\pi}{2}$	1
π	0
$\frac{3\pi}{2}$	-1
2π	0

Approximate π to at least 6 decimal places in the following computations.

(*a*) Compute the quartic Lagrange polynomial $L_4(x)$ that passes through all 5 of these points.

(*b*) Estimate $\sin\frac{\pi}{6}$ by computing $L_4(\frac{\pi}{6})$.

(*c*) Compute the clamped cubic spline interpolant for these 5 points, using the obvious condition $S'(0) = 1$ and $S'(2\pi) = 1$. (We can compute those values because we know the derivative of the sine function.)

(*d*) Estimate $\sin\frac{\pi}{6}$ by computing $S(\frac{\pi}{6})$.

(*e*) Which approximation is more accurate?

Answer: (*a*) We need not write out the terms which reduce to 0, so we get

$$\begin{aligned} L_4(x) &= \frac{(x-0)(x-\pi)(x-\frac{3\pi}{2})(x-2\pi)}{(\frac{\pi}{2}-0)(\frac{\pi}{2}-\pi)(\frac{\pi}{2}-\frac{3\pi}{2})(\frac{\pi}{2}-2\pi)} - \frac{(x-0)(x-\frac{\pi}{2})(x-\pi)(x-2\pi)}{(\frac{3\pi}{2}-0)(\frac{3\pi}{2}-\frac{\pi}{2})(\frac{3\pi}{2}-\pi)(\frac{3\pi}{2}-2\pi)} \\ &\approx +0.086004x^3 - 0.810570x^2 + 1.697653x \end{aligned}$$

(*b*) We can then approximate $L_4(\frac{\pi}{6}) \approx 0.679008$.

(*c*) We have $a_0 = a_2 = a_4 = 0$, $a_1 = 1$, and $a_3 = -1$. We also have $h_0 = h_1 = h_2 = h_3 = 1.570796$. The matrix equation is:

$$\begin{bmatrix} 3.141593 & 1.570796 & 0.000000 & 0.000000 & 0.000000 \\ 1.570796 & 6.283185 & 1.570796 & 0.000000 & 0.000000 \\ 0.000000 & 1.570796 & 6.283185 & 1.570796 & 0.000000 \\ 0.000000 & 0.000000 & 1.570796 & 6.283185 & 1.570796 \\ 0.000000 & 0.000000 & 0.000000 & 1.570796 & 3.141593 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -0.049187 \\ -0.595630 \\ +0.000000 \\ +0.595630 \\ +0.049187 \end{bmatrix}$$

This has solution $(c_0, c_1, c_2, c_3, c_4) = (-0.049187, -0.595630, 0.000000, 0.595630, 0.049187)$, and then we have

$$\begin{aligned} (b_0, b_1, b_2, b_3) &= (1.000000, -0.012877, -0.948491, -0.012877) \\ (d_0, d_1, d_2, d_3) &= (-0.115959, 0.126397, 0.126397, -0.115959) \end{aligned}$$

This gives:

$$S(x) = \begin{cases} 0 + x - 0.049187x^2 - 0.115959x^3 & 0 \le x \le \frac{\pi}{2} \\ 1 - 0.012877(x-\frac{\pi}{2}) - 0.595630(x-\frac{\pi}{2})^2 + 0.126397(x-\frac{\pi}{2})^3 & \frac{\pi}{2} \le x \le \pi \\ 0 - 0.948491(x-\pi) + 0.126397(x-\pi)^3 & \pi \le x \le \frac{3\pi}{2} \\ -1 - 0.012877(x-\frac{3\pi}{2}) + 0.595630(x-\frac{3\pi}{2})^2 - 0.115959(x-\frac{3\pi}{2})^3 & \frac{3\pi}{2} \le x \le 2\pi \end{cases}$$

This can be rewritten as:

$$S(x) = \begin{cases} -0.115959x^3 - 0.049187x^2 + x & 0 \le x \le \frac{\pi}{2} \\ 0.126397x^3 - 1.191262x^2 + 2.793966x - 0.939319 & \frac{\pi}{2} \le x \le \pi \\ 0.126397x^3 - 1.191264x^2 + 2.793974x - 0.939328 & \pi \le x \le \frac{3\pi}{2} \\ -0.115959x^3 + 2.234962x^2 - 13.351726x + 24.422271 & \frac{3\pi}{2} \le x \le 2\pi \end{cases}$$

(*d*) We have $S(\frac{\pi}{6}) = 0.493464$.

(*e*) Because the value of $\sin\frac{\pi}{6} = 0.5$, we can see that the cubic spline gives a considerably more accurate value than does the Lagrange polynomial. Even the free cubic spline does better: it gives a value of 0.481481, which is still much closer to 0.5 than 0.679008.