MT414: Numerical Analysis
Homework 6
Answers

1. Suppose that we have the following values for a function $f(x)$:

x	$f(x)$
2.1	1.5602
2.2	1.4905
2.3	1.4324
2.4	1.3833
2.5	1.3415
2.6	1.3055

Use formulas from the text and class to estimate as accurately as possible the values of $f'(x)$ for $x = 2.1, 2.2, \ldots, 2.6$.

Answer: The relevant formulæ are

$$f'(x_0) \approx \frac{1}{12h}[f(x_0-2h) - 8f(x_0-h) + 8f(x_0+h) - f(x_0+2h)] \tag{4.6}$$

$$f'(x_0) \approx \frac{1}{12h}[-25f(x_0) + 48f(x_0+h) - 36f(x_0+2h) + 16f(x_0+3h) - 3f(x_0+4h)] \tag{4.7}$$

If we apply (4.7) with $x_0 = 2.1$ and $h = 0.1$, we get -0.7659, while using $x_0 = 2.2$ and $h = 0.1$ gives -0.6322. We can use (4.6), with $x_0 = 2.3$ and $h = 0.1$ to get -0.5324, and using (4.6) with $x_0 = 2.4$ and $h = 0.1$ gives -0.4518. Finally, using (4.5) with $x_0 = 2.5$ and $h = -0.1$ gives -0.3849, while setting $x_0 = 2.6$ and $h = -0.1$ gives -0.3355.

2. Suppose that for some fixed values of x_0 and h, we know $f(x_0-h)$, $f(x_0)$, $f(x_0+h)$, and $f(x_0+2h)$. Derive a 4-point formula to estimate $f'(x_0)$ to $O(h^3)$.

Answer: We have

$$\begin{aligned}
f(x_0-h) &= f(x_0) - hf'(x_0) + \frac{h^2}{2}f''(x_0) - \frac{h^3}{6}f^{(3)}(x_0) + \frac{h^4}{24}f^{(4)}(\xi_1) \\
f(x_0+h) &= f(x_0) + hf'(x_0) + \frac{h^2}{2}f''(x_0) + \frac{h^3}{6}f^{(3)}(x_0) + \frac{h^4}{24}f^{(4)}(\xi_2) \\
f(x_0+2h) &= f(x_0) + 2hf'(x_0) + 2h^2f''(x_0) + \frac{4h^3}{3}f^{(3)}(x_0) + \frac{16h^4}{24}f^{(4)}(\xi_3)
\end{aligned}$$

Multiply the first equation by A, the second by B, the third by C, and add. We would like to eliminate the terms involving $f''(x_0)$ and $f^{(3)}(x_0)$. The result is two equations:

$$\begin{aligned}
\frac{A}{2} + \frac{B}{2} + 2C &= 0 \\
-\frac{A}{6} + \frac{B}{6} + \frac{4C}{3} &= 0
\end{aligned}$$

Multiply the second by 3, yielding $-\frac{A}{2} + \frac{B}{2} + 4C = 0$, and add to the first one to get $B + 6C = 0$. So take $C = 1$, and then $B = -6$ and $A = 2$. Thus, we have

$$2f(x_0-h) - 6f(x_0+h) + f(x_0+2h) = -3f(x_0) - 6hf'(x_0) + \frac{h^4}{24}\left(2f^{(4)}(\xi_1) - 6f^{(4)}(\xi_2) + 16f^{(4)}(\xi_3)\right)$$

Solving for $f'(x_0)$, we have

$$f'(x_0) = \frac{-2f(x_0-h) - 3f(x_0) + 6f(x_0+h) - f(x_0+2h)}{6h} + \frac{h^3}{144}\left(2f^{(4)}(\xi_1) - 6f^{(4)}(\xi_2) + 16f^{(4)}(\xi_3)\right)$$

More complicated methods allow one to show that

$$\frac{h^3}{144}\left(2f^{(4)}(\xi_1)-6f^{(4)}(\xi_2)+16f^{(4)}(\xi_3)\right)=\frac{h^3}{72}\left(f^{(4)}(\xi_1)-3f^{(4)}(\xi_2)+8f^{(4)}(\xi_3)\right)$$
$$=\frac{h^3}{12}\left(\frac{f^{(4)}(\xi_1)-3f^{(4)}(\xi_2)+8f^{(4)}(\xi_3)}{6}\right)=\frac{h^3}{12}f^{(4)}(\xi).$$

so the eventual answer is:

$$f'(x_0)=\frac{1}{6h}\left(-2f(x_0-h)-3f(x_0)+6f(x_0+h)-f(x_0+2h)\right)+\frac{h^3}{12}f^{(4)}(\xi).$$

3. Suppose that $N(h)$ is an approximation to a quantity M for every $h>0$, and that

$$M=N(h)+K_1h^2+K_2h^4+K_3h^6+\cdots,$$

for some constants $K_1, K_2, K_3, \ldots$. Use the values $N(h)$, $N(h/3)$, and $N(h/9)$ to produce an $O(h^6)$ approximation for M.

Answer: We have

$$M=N(h)+K_1h^2+K_2h^4+K_3h^6+\cdots$$
$$M=N(h/3)+K_1\frac{h^2}{9}+K_2\frac{h^4}{81}+K_3\frac{h^6}{729}+\cdots$$
$$M=N(h/9)+K_1\frac{h^2}{81}+K_2\frac{h^4}{6561}+K_3\frac{h^6}{531441}+\cdots$$

Multiply the first equation by A, the second by B, and the third by C. Adding and canceling the K_1 and K_2 terms yields the equations

$$A+\frac{B}{9}+\frac{C}{81}=0$$
$$A+\frac{B}{81}+\frac{C}{6561}=0$$

Subtracting gives $\frac{8B}{81}+\frac{80C}{6561}=0$. Multiply by by 6561, and we have $648B+80C=0$, or $81B+10C=0$. Set $C=-81$, $B=10$, and $A=-\frac{1}{9}$. Therefore,

$$\left(-\frac{1}{9}+10-81\right)M=-\frac{N(h)}{9}+10N(h/3)-81N(h/9)+O(h^6),$$

or

$$M=\frac{1}{640}\left(729N(h/9)-90N(h/3)+N(h)\right)+O(h^6).$$

4. (a) Show that

$$\lim_{h\to 0}\left(\frac{2+h}{2-h}\right)^{1/h}=e.$$

(b) Compute approximations to e using the formula

$$N(h)=\left(\frac{2+h}{2-h}\right)^{1/h},$$

for $h=0.4$, 0.2, and 0.1.

(c) Assuming that $e=N(h)+K_1h+K_2h^2+K_3h^3+\cdots$. Use extrapolation to compute an $O(h^3)$ approximation to e with $h=0.4$.

(d) Show that $N(-h)=N(h)$.

(e) Use part (d) to show that $K_1 = K_3 = K_5 = \cdots = 0$ in the formula

$$e = N(h) + K_1 h + K_2 h^2 + K_3 h^3 + K_4 h^4 + K_5 h^5 + \cdots,$$

so that the formula reduces to

$$e = N(h) + K_2 h^2 + K_4 h^4 + K_6 h^6 + \cdots.$$

(f) Use the result of part (e) and an extrapolation to compute an $O(h^6)$ approximation to e with $h = 0.4$.

Answer: (a) Let $C = \left(\frac{2+h}{2-h}\right)^{1/h}$. Rather than evaluate $\lim_{h\to 0} C$, we evaluate $\lim_{h\to 0} \log C$. Note that $\log C = (\frac{1}{h})(\log(2+h) - \log(2-h))$, so we have

$$\lim_{h\to 0} \log C = \lim_{h\to 0} \frac{\log(2+h) - \log(2-h)}{h} = \lim_{h\to 0} \frac{\frac{1}{2+h} + \frac{1}{2-h}}{1} = \lim_{h\to 0} \frac{4}{(2+h)(2-h)} = 1.$$

Therefore, $\log \lim_{h\to 0} C = \lim_{h\to 0} \log C = 1$, so $\lim_{h\to 0} C = e$.

(b) Substitution of $h = 0.4$, 0.2, and 0.1 yields 2.7557, 2.7274, and 2.7206 respectively.

(c) We could apply the formulas in the book, but it's fun to do this directly:

$$\begin{aligned} e &= N(0.4) + 0.4K_1 + 0.16K_2 + K_3 0.4^3 + \cdots \\ e &= N(0.2) + 0.2K_1 + 0.04K_2 + K_3 0.2^3 + \cdots \\ e &= N(0.1) + 0.1K_1 + 0.01K_2 + K_3 0.1^3 + \cdots \end{aligned}$$

Multiply the first equation by A, the second by B, and the third by C. Add, and set the terms involving K_1 and K_2 to 0, yielding the 2 equations:

$$\begin{aligned} 0.40A + 0.20B + 0.10C &= 0 \\ 0.16A + 0.04B + 0.01C &= 0 \end{aligned}$$

Multiply the first equation by 10 and the second by 100 to eliminate fractions, and then we have

$$\begin{aligned} 4A + 2B + C &= 0 \\ 16A + 4B + C &= 0 \end{aligned}$$

Subtracting gives $12A + 2B = 0$, so set $B = -6$, and $A = 1$, and then $C = 8$. Adding up the equations now gives

$$3e = N(0.4) - 6N(0.2) + 8N(0.1) + O(h^3),$$

or $e \approx (8N(0.1) - 6N(0.2) + N(0.4))/3$. This gives $e \approx 2.7185$.

(d) We have

$$N(-h) = \left(\frac{2-h}{2+h}\right)^{\frac{-1}{h}} = \left(\left(\frac{2-h}{2+h}\right)^{-1}\right)^{\frac{1}{h}} = \left(\frac{2+h}{2-h}\right)^{\frac{1}{h}} = N(h).$$

(e) We have

$$\begin{aligned} e &= N(+h) + K_1 h + K_2 h^2 + K_3 h^3 + K_4 h^4 + K_5 h^5 + \cdots \\ e &= N(-h) - K_1 h + K_2 h^2 - K_3 h^3 + K_4 h^4 - K_5 h^5 + \cdots \end{aligned}$$

so adding, using the fact that $N(h) = N(-h)$, and dividing by 2 yields

$$e = N(h) + K_2 h^2 + K_4 h^4 + K_6 h^6 + \cdots.$$

(f) So now we have

$$e = N(0.4) + 0.16K_2 + 0.0256K_4 + \cdots$$
$$e = N(0.2) + 0.04K_2 + 0.0016K_4 + \cdots$$
$$e = N(0.1) + 0.01K_2 + 0.0001K_4 + \cdots$$

Our usual procedure, followed by multiplication to eliminate decimals, gives the equations

$$16A + \ \ 4B + C = 0$$
$$256A + 16B + C = 0$$

and subtraction gives $240A + 12B = 0$. Set $B = -20$, $A = 1$, and $C = 64$, and we have

$$45e = N(0.4) - 20N(0.2) + 64N(0.1) + O(h^6),$$

or $e \approx \frac{1}{45}(64N(0.1) - 20N(0.2) + N(0.4)) = 2.7183$. Note that this answer is correct to the accuracy to which we have worked. In fact, to 7 places we have 2.7182824, so this method yields a quite accurate estimate.

5. Show that

$$\int_a^b f(x)\,dx = 2hf(x_0) + \frac{h^3}{3}f''(\xi),$$

where $b - a = 2h$, $x_0 = a + h$, and $\xi \in (a, b)$.

Answer: To avoid typing subscripts repeatedly, I will let $y = a+h$, and then we are estimating $\displaystyle\int_{y-h}^{y+h} f(x)\,dx$. Fix y, and set

$$E(h) = \int_{y-h}^{y+h} f(x)\,dx - 2hf(y) = \int_y^{y+h} f(x)\,dx - \int_y^{y-h} f(x)\,dx - 2hf(y).$$

Differentiate, and we have

$$E'(h) = f(y+h) + f(y-h) - 2f(y)$$

by using the Fundamental Theorem of Calculus (and the chain rule). Differentiate again, and we have

$$E''(h) = f'(y+h) - f'(y-h) = 2h\left(\frac{f'(y+h) - f'(y-h)}{2h}\right) = 2hf''(\xi_1).$$

Integration and an application of the Mean Value Theorem for Integrals gives

$$E'(h) = h^2 f''(\xi_2).$$

Integrate again, again applying the Mean Value Theorem for Integrals, and we have

$$E(h) = \frac{h^3 f''(\xi)}{3}.$$