1. Suppose that I roll 7 identical ordinary (cubical) dice.
   (a) What is the probability of exactly 1 pair of matching numbers on the 7 dice?
   (b) What is the probability of 3 pairs of distinct matching numbers on the dice? Distinct means that the 3 pairs show 3 different numbers, so that we would count 1122334, but not 1111223, as having 3 distinct matching pairs.
   (c) What is the probability that all 6 numbers from 1 to 6 are visible on the 7 dice?

   **Answer:** (a) Pick a number for the pair: 6 options. Pick the place for the pair of matching dice: \( \binom{7}{2} \) options. Pick the numbers on the remaining 5 dice: 5! options. The probability is therefore
   \[
   \frac{6 \cdot \binom{7}{2} \cdot 5!}{6^7} = \frac{35}{648} \approx 0.0540.
   \]

   (b) Pick the number and location for the first pair: 6\( \binom{7}{2} \) options. Pick the number and location for the second pair: 5\( \binom{5}{2} \) options. Pick the number and location for the third pair: 4\( \binom{3}{2} \) options. Pick the remaining number: 3 options. Then divide by 3! because there is no first, second, and third pair, only 3 unordered pairs. (Equivalently, pick the numbers on the pairs using \( \binom{6}{3} \) rather than 6 \cdot 5 \cdot 4.) The probability is
   \[
   \frac{6 \cdot \binom{7}{2} \cdot 5 \cdot \binom{5}{2} \cdot 4 \cdot \binom{3}{2} \cdot 3!}{6^7} = \frac{175}{1296} \approx 0.1350.
   \]

   (c) This is exactly the same as the probability of a single pair, so the answer is the same as above. However, we can also do the problem using the Inclusion–Exclusion Principle, and that approach is more helpful when generalizing this question. Let \( E_i \) be the event that the number \( i \) is not showing on any of the 7 dice. We need to compute \( 1 - P(E_1 \cup E_2 \cup \cdots \cup E_6) \).

   We have
   \[
   P(\cup E_k) = \sum P(E_k) - \sum P(E_k E_j) + \cdots = 6P(E_k) - \binom{6}{2} P(E_k E_j) + \binom{6}{3} P(E_k E_j E_i) - \cdots
   \]
   \[
   = 6 \left( \frac{5}{6} \right)^7 - 15 \left( \frac{4}{6} \right)^7 + 20 \left( \frac{3}{6} \right)^7 - 15 \left( \frac{2}{6} \right)^7 + 6 \left( \frac{1}{6} \right)^7 = \frac{613}{648} \approx 0.9460.
   \]

   Note that as promised \( 1 - \frac{613}{648} = \frac{35}{648} \).

2. Ninety-eight percent of all babies survive delivery. However, 15% of all births involve a procedure called a Cæsarean section (usually called a C-section). When a C-section is performed, the baby survives 96% of the time. Suppose a randomly chosen pregnant woman gives birth without a C-section. What is the probability that the baby survives?

   **Answer:** Let
   \[
   C = \{ \text{C-section performed} \} \]
\[
S = \{\text{Baby survives}\}
\]

We have \(P(S) = 0.98\), \(P(C) = 0.15\), and \(P(S|C) = 0.96\). We know that \(P(S) = P(S|C)P(C) + P(S|C^c)P(C^c)\), so \(0.98 = 0.96 \cdot 0.15 + 0.85P(S|C^c)\). Solving, we have \(P(S|C^c) = 0.9835\).

3. In a certain community, 36% of all families own a dog, 30% of all families own a cat, and 22% of the dog owners also own a cat.
   
   (a) What is the probability that a family owns both a dog and a cat?
   
   (b) What is the probability that a randomly chosen family of cat owners also owns a dog?

   \textbf{Answer:} Let

   \[
   D = \{\text{family owns a dog}\}
   \]
   \[
   C = \{\text{family owns a cat}\}
   \]

   We are told \(P(D) = 0.36\), \(P(C) = 0.30\), and \(P(C|D) = 0.22\).
   
   (a) \(P(D|C) = P(C|D)P(D)/P(C) = 0.0792/0.30 = 0.2640\). Therefore, 26.4% of cat owners also own dogs.

4. Suppose that in a standard game of bridge, North–South have 9 spades between them, so that East–West have 4 spades.
   
   (a) What is the probability that both East and West have 2 spades?
   
   (b) What is the probability that either East or West has exactly 3 spades?
   
   (c) What is the probability that either East or West has 4 spades?

   \textbf{Answer:} Note that EW have 4 spades between them and 22 other cards.
   
   (a) The probability that East has exactly 2 spades is
   
   \[
   \frac{\binom{4}{2}\binom{22}{11}}{\binom{26}{13}} = \frac{234}{575} \approx 0.4070.
   \]

   (b) The probability that East has exactly 3 spades is
   
   \[
   \frac{\binom{4}{3}\binom{22}{10}}{\binom{26}{13}} = \frac{143}{575} \approx 0.2487.
   \]

   This is the same as the probability that West has exactly 3 spades, so the answer to the problem is \(\frac{143}{575} \approx 0.4974\).

   (c) The probability that East has exactly 4 spades is
   
   \[
   \frac{\binom{4}{4}\binom{22}{9}}{\binom{26}{13}} = \frac{11}{230} \approx 0.0478.
   \]

   This is the same as the probability that West has exactly 4 spades, so the answer to the problem is \(\frac{11}{115} \approx 0.0957\).
Note that the 3 probabilities sum to 1, as they must.

5. There are 3 coins in a box. One of them is a fair coin, the second is a two-headed coin, and the third coin is weighted so that it comes up heads 75% of the time.

(a) Suppose that 1 of the 3 coins is chosen at random and flipped, and it shows heads. What is the probability that the two-headed coin was chosen?

(b) Suppose instead that 1 of the 3 coins is chosen at random and flipped, and it shows tails. What is the probability that the fair coin was chosen?

Answer: Let

\[ F = \{ \text{Fair coin is chosen} \} \]
\[ T = \{ \text{Two-headed coin is chosen} \} \]
\[ W = \{ \text{Weighted coin is chosen} \} \]
\[ H = \{ \text{Coin comes up heads} \} \]

We have \( P(F) = P(T) = P(W) = \frac{1}{3} \), \( P(H|F) = \frac{1}{2} \), \( P(H|T) = 1 \), and \( P(H|W) = \frac{3}{4} \). Therefore, \( P(H) = \frac{1}{2} + \frac{1}{3} + \frac{3}{4} = \frac{13}{12} \).

(a) We have \( P(T|H) = P(H|T)P(T)/P(H) = \frac{1}{3} \frac{1}{4} = \frac{4}{3} \).

(b) We have \( P(F|H^c) = P(H|F)P(F)/P(H^c) = \frac{1}{2} \frac{1}{3} \frac{1}{4} = \frac{2}{3} \).

6. An urn contains 5 white and 10 black balls. A fair die is rolled, and that number of balls is randomly selected without replacement from the urn.

(a) What is the probability that all of the balls selected are white?

(b) What is the conditional probability that the die landed on 3 if we are told that all of the balls selected were white?

Answer: Let

\[ W = \{ \text{All selected balls are white} \} \]
\[ E_k = \{ \text{Die shows } k \}, \quad k = 1, 2, \ldots, 6 \]

Then

\[
P(E_k) = \frac{1}{6} \\
P(W|E_1) = \frac{\binom{5}{1}}{\binom{15}{1}} = \frac{1}{3} \\
P(W|E_2) = \frac{\binom{5}{2}}{\binom{15}{2}} = \frac{2}{21} \\
P(W|E_3) = \frac{\binom{5}{3}}{\binom{15}{3}} = \frac{2}{91} \\
P(W|E_4) = \frac{\binom{5}{4}}{\binom{15}{4}} = \frac{1}{273}
\]
P(W|E_5) = \frac{(\binom{5}{5})}{(\binom{15}{5})} = \frac{1}{3003}

P(W|E_6) = 0

(a) We have P(W) = \frac{1}{6}(\frac{1}{3} + \frac{2}{27} + \frac{2}{7}) + \frac{1}{273} + \frac{1}{3003} = \frac{5}{66} \approx 0.0758.

(b) We have P(E_3|W) = P(W|E_3)P(E_3)/P(W) = (\frac{2}{3})(\frac{1}{6})/\frac{5}{66} = \frac{22}{435} \approx 0.0484.

7. Prostate cancer is a common type of cancer in men. A test measuring PSA (Prostate Specific Antigen) is commonly employed but is unreliable. The probability that a non-cancerous man will have an elevated PSA is approximately 0.135, and the probability that a cancerous man will have an elevated PSA is approximately 0.268. Suppose that based on family history and a physical examination, a physician is 70% certain that a man has prostate cancer, and measures his PSA.

(a) If the test indicates elevated PSA level, what is the probability that the man has cancer?

(b) If the test indicates normal PSA level, what is the probability that the man has cancer?

Answer: Let

\[ C = \{ \text{Patient has prostate cancer} \} \]
\[ E = \{ \text{Patient has elevated PSA} \} \]

We are given P(E|C^c) = 0.135, P(E|C) = 0.268, and P(C) = 0.7. We compute P(E) = P(E|C^c)P(C^c) + P(E|C)P(C) = 0.2281 and P(E^c) = 1 - P(E) = 0.7719.

(a) We have P(C|E) = P(E|C)P(C)/P(E) = 0.8224.

(b) We have P(C|E^c) = P(E^c|C)P(C)/P(E^c) = (1 - P(E|C))0.7/0.7719 = 0.6638.

Note that although a positive test increases the probability that the man has cancer quite a bit, a negative test does not decrease the probability by a substantial amount.

You might be interested in repeating this calculation if P(C) = 0.2.