

Mathematics 805
Examination 1
Answers

1. (20 points) Suppose that A and B are subsets of a metric space. Prove or give a counterexample:

- (a) $(A \cap B)^\circ = A^\circ \cap B^\circ$.
- (b) $(A \cap B)^- = A^- \cap B^-$.

Answer: (a) This is true. Suppose first that $x \in (A \cap B)^\circ$. Then there is some $\epsilon > 0$ so that $B_\epsilon(x) \subset A \cap B$. Therefore, $B_\epsilon(x) \subset A$ and $B_\epsilon(x) \subset B$, so $x \in A^\circ$ and $x \in B^\circ$.

Suppose on the other hand that $x \in A^\circ \cap B^\circ$. Then there is some $\epsilon_1 > 0$ so that $B_{\epsilon_1}(x) \subset A$ and some $\epsilon_2 > 0$ so that $B_{\epsilon_2}(x) \subset B$. Let $\epsilon = \min(\epsilon_1, \epsilon_2)$, and then $B_\epsilon(x) \subset A \cap B$.

(b) This is false. Let $A = (0, 1)$, $B = (1, 2)$, and then $A^- = [0, 1]$, $B^- = [1, 2]$, $(A \cap B)^- = \emptyset^- = \emptyset$, and $A^- \cap B^- = \{1\}$.

2. (20 points) (a) Define *compact*.

- (b) Using only your definition, prove that the union of 2 compact sets is compact.

Answer: (a) A set K is *compact* if every open cover of K has a finite subcover.

(b) Suppose that A and B are both compact. Suppose that $\{U_\lambda\}_{\lambda \in \Lambda}$ is an open cover of $A \cup B$. Then $\{U_\lambda\}_{\lambda \in \Lambda}$ is an open cover of A , so there is a finite subcover U_1, U_2, \dots, U_n of A . But also $\{U_\lambda\}_{\lambda \in \Lambda}$ is an open cover of B , so there is a finite subcover U'_1, U'_2, \dots, U'_m of B . Then $U_1, U_2, \dots, U_n, U'_1, U'_2, \dots, U'_m$ is a finite subcover of $A \cup B$.

3. (20 points) Suppose that A and B are subsets of a metric space. Prove or give a counterexample:

- (a) If $A \subset B$, then $A^- \subset B^-$.
- (b) If $A \subset B$, then $A^\circ \subset B^\circ$.

Answer: (a) This is true. Suppose that $x \in A^-$. Then for every $\epsilon > 0$, $B_\epsilon(x) \cap A \neq \emptyset$. Because $A \subset B$, we know that $B_\epsilon(x) \cap B \neq \emptyset$. Therefore, $x \in B^-$.

(b) This is true. Suppose that $x \in A^\circ$. Then there is some $\epsilon > 0$ for which $B_\epsilon(x) \subset A$. Because $A \subset B$, we have $B_\epsilon(x) \subset B$. Therefore, $x \in B^\circ$.

4. (20 points) (a) Define *connected* set.

- (b) Show that \mathbf{Q} is not a connected set.

Answer: (a) A set C is connected if it is *not* possible to write $C = U \cup V$, with $U \cap V = \emptyset$, with U and V open and nonempty.

- (b) Write $\mathbf{Q} = (-\infty, \sqrt{2}) \cup (\sqrt{2}, \infty)$.

5. (20 points) (a) Let M_1 and M_2 be metric spaces, and let $f : M_1 \rightarrow M_2$. State the definition of *uniformly continuous*.

(b) Suppose that D is a subset of a *compact* metric space M , and $f : D \rightarrow \mathbf{R}$ is uniformly continuous. Prove that $f(D)$ is bounded, preferably by applying several theorems.

Answer: (a) For any $\epsilon > 0$, there is a $\delta > 0$ so that if $x, y \in M_1$ and $d(x, y) < \delta$, then $d(f(x), f(y)) < \epsilon$.

(b) Because $f : D \rightarrow \mathbf{R}$ is uniformly continuous, we know that f has a continuous extension $\tilde{f} : D^- \rightarrow \mathbf{R}$. Because D^- is a closed subset of the compact set M , we know that D^- is compact. Because \tilde{f} is continuous, we know that $\tilde{f}(D^-)$ is compact and hence bounded. Finally, $f(D) \subset \tilde{f}(D^-)$, so $f(D)$ is bounded.