Mathematics 805
Final Examination
May 6, 2009

Please do all of your work in the blue booklets. Please work clearly and neatly, and label your answers. You do not need to do the problems in order. No credit will be given for answers without explanations.

The problems are not arranged in order of increasing difficulty, so you might want to read all of them before beginning.

1. *(5 points)* State the Weierstrass M-test.

2. *(15 points)* (a) State Lebesgue's Dominated Convergence Theorem.
   (b) State Lebesgue's Monotone Convergence Theorem.
   (c) Define

$$f_n(x) = \begin{cases} \dfrac{1}{n} & x \in [0, n] \\ 0 & \text{otherwise} \end{cases}$$

Note that $\displaystyle\int f_n(x)\,dx = 1$, but $\displaystyle\lim_{n\to\infty} f_n(x) = 0$. Why does this contradict neither the Dominated nor the Monotone Convergence Theorems?

3. *(10 points)* Suppose that K is a compact metric space, and $g : K \to \mathbf{R}$ a continuous function, with $g(x) > 0$ for all $x \in K$. Suppose further that $g_n : K \to \mathbf{R}$ is a sequence of continuous functions converging uniformly to g on K. Show that there is some integer N so that if $n > N$, then $g_n(x) > 0$ for all $x \in K$.

4. *(15 points)* (a) Let (M_1, d_1) and (M_2, d_2) be metric spaces, and $f : M_1 \to M_2$ a function. Define what is meant by "f is uniformly continuous."
   (b) Suppose that (M_1, d_1) and (M_2, d_2) are metric spaces, and $f : M_1 \to M_2$ is uniformly continuous. Suppose that $A, B \subset M_1$ with $d_1(A, B) = 0$. Show that $d_2(f(A), f(B)) = 0$.
   (c) Give an example to show that it is possible for $f : M_1 \to M_2$ to be continuous, with $d_1(A, B) = 0$ and $d_2(f(A), f(B)) \neq 0$.

5. *(12 points)* Show that

$$\int_1^\infty \sin(x^2)\,dx$$

can be defined as an improper Riemann integral but not as a Lebesgue integral. *Hint:* Let $t = x^2$, and imitate the proof that $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx$ can be defined as an improper Riemann integral but not a Lebesgue integral.

6. *(5 points)* As usual, we define

$$\frac{te^{tx}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n$$

and $B_n = B_n(0)$. (Recall that $B_0(x) = 1$, $B_1(x) = x - \frac{1}{2}$, and $B_2(x) = x^2 - x + \frac{1}{6}$.) Show that $B_n(\frac{1}{2}) = (2^{1-n} - 1)B_n$.

7. *(10 points)* Suppose that $f \in C^\infty$. Let $m \geq 1$ be an integer. Prove using induction on m that

$$f(0) = \int_0^1 f(x)\,dx + \sum_{k=1}^{m} \frac{B_k}{k!}\left(f^{(k-1)}(1) - f^{(k-1)}(0)\right) - (-1)^m \int_0^1 \frac{B_m(x)}{m!} f^{(m)}(x)\,dx$$

8. *(28 points)* Here is yet another way to compute $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx$. Recall the following facts, some of which might be needed to do this problem:

$$\begin{aligned}
D_n(x) &:= \sum_{k=-n}^{n} e^{ikx} \\
D_n(x) &= \frac{\sin(n+\frac{1}{2})x}{\sin(\frac{x}{2})} \\
K_n(x) &:= \frac{1}{n+1}\sum_{k=0}^{n} D_k(x) \\
K_n(x) &= \left(\frac{1}{n+1}\right)\left(\frac{1-\cos(n+1)x}{1-\cos x}\right) \\
&= \left(\frac{1}{n+1}\right)\left(\frac{\sin(n+1)x/2}{\sin(x/2)}\right)^2 \\
\frac{1}{\pi}\int_0^\pi D_n(x)\,dx &= \frac{1}{\pi}\int_0^\pi K_n(x)\,dx = 1 \\
1 - \cos x &= 2\sin^2(x/2)
\end{aligned}$$

(a) Suppose that g is Riemann-integrable on $[a, b]$ with $|g| \leq M$. Show that

$$\left|\int_a^b g(x)\left(\frac{1-\cos kx}{k}\right)\,dx\right| \leq \frac{2(b-a)M}{k}$$

and conclude that

$$\lim_{k\to\infty}\int_a^b g(x)\left(\frac{1-\cos kx}{k}\right)\,dx = 0.$$

(b) Show that

$$\int_0^\pi \left(\frac{1}{k}\right)\left(\frac{1-\cos kx}{4\sin^2\frac{x}{2}}\right)\,dx = \frac{\pi}{2}$$

(c) Let

$$f(x) = \frac{1}{x^2} - \frac{1}{4\sin^2\frac{x}{2}}, \qquad 0 < x \leq \pi$$

Show that it is possible to define $f(0)$ so that $f(x)$ is continuous and bounded on $[0, \pi]$.

(d) Show that

$$\lim_{k\to\infty} \int_0^\pi f(x) \left(\frac{1 - \cos kx}{k} \right) dx = 0.$$

(e) Show that

$$\lim_{k\to\infty} \int_0^\pi \frac{1}{x^2} \left(\frac{1 - \cos kx}{k} \right) dx = \frac{\pi}{2}$$

and then

$$\lim_{k\to\infty} \int_0^\pi \frac{\sin^2(kx/2)}{kx^2} dx = \frac{\pi}{4}.$$

(f) Let $y = \frac{kx}{2}$ and conclude that

$$\frac{\pi}{2} = \lim_{k\to\infty} \int_0^{k\pi/2} \frac{\sin^2 x}{x^2} dx = \int_0^\infty \frac{\sin^2 x}{x^2} dx.$$

(g) Finally, integrate by parts to show that

$$\int_0^\infty \frac{\sin^2 x}{x^2} dx = \int_0^\infty \frac{\sin x}{x} dx.$$