1. (5 points) State the Weierstrass M-test.
Answer: Suppose that $A \subset \mathbb{R}$, and $f_n : A \to \mathbb{R}$. Suppose further that $|f_n(x)| \leq M_n$ for all $x \in A$, and that $\sum M_n$ converges. Then $\sum f_n(x)$ converges uniformly on $A$.

2. (15 points) (a) State Lebesgue’s Dominated Convergence Theorem.
   (b) State Lebesgue’s Monotone Convergence Theorem.
   (c) Define
   
   $$f_n(x) = \begin{cases} \frac{1}{n} & x \in [0,n] \\ 0 & \text{otherwise} \end{cases}$$

   Note that $\int f_n(x) \, dx = 1$, but $\lim_{n \to \infty} f_n(x) = 0$. Why does this contradict neither the Dominated nor the Monotone Convergence Theorems?
Answer: (a) Suppose that $f_n \in L^1$ and $\lim f_n = f$ almost everyone. Suppose further that $|f_n| \leq g$ almost everywhere, with $g \in L^1$. Then $f \in L^1$ and $\int f = \lim \int f_n$.
(b) Suppose that $f_n \in L^1$ is a monotone sequence, and suppose further that $\int f_n$ is bounded. Then $f_n$ converges almost everywhere to a function $f \in L^1$, and $\int f = \lim \int f_n$.
(c) This example does not violate the Dominated Convergence Theorem, because there is no function $g \in L^1$ with $|f_n| \leq g$. The example does not violate the Monotone Convergence Theorem because the sequence $f_n$ is not monotone.

3. (10 points) Suppose that $K$ is a compact metric space, and $g : K \to \mathbb{R}$ a continuous function, with $g(x) > 0$ for all $x \in K$. Suppose further that $g_n : K \to \mathbb{R}$ is a sequence of continuous functions converging uniformly to $g$ on $K$. Show that there is some integer $N$ so that if $n > N$, then $g_n(x) > 0$ for all $x \in K$.
Answer: Because $K$ is compact, we know that $g$ attains its minimum value at some $x \in K$. In particular, we can find some $\epsilon$ so that $g(x) \geq \epsilon$ for all $x \in K$.

   Because $g_n \to g$ uniformly, we can find an integer $N$ so that if $n \geq N$, then $|g(x) - g_n(x)| \leq \epsilon/2$ for all $x \in K$. This implies that $g_n(x) \geq \epsilon/2$ for all $x \in K$, so $g_n(x) > 0$.

4. (15 points) (a) Let $(M_1, d_1)$ and $(M_2, d_2)$ be metric spaces, and $f : M_1 \to M_2$ a function. Define what is meant by “$f$ is uniformly continuous.”
   (b) Suppose that $(M_1, d_1)$ and $(M_2, d_2)$ are metric spaces, and $f : M_1 \to M_2$ is uniformly continuous. Suppose that $A, B \subset M_1$ with $d_1(A,B) = 0$. Show that $d_2(f(A), f(B)) = 0$.
   (c) Give an example to show that it is possible for $f : M_1 \to M_2$ to be continuous, with $d_1(A,B) = 0$ and $d_2(f(A), f(B)) \neq 0$.
Answer: (a) Given any $\epsilon > 0$, there is a $\delta > 0$ so that if $d_1(x,y) < \delta$, then $d_2(f(x), f(y)) < \epsilon$.
(b) Given any $\epsilon > 0$, we can find $\delta > 0$ so that if $d_1(x,y) < \delta$, then $d_2(f(x), f(y)) < \epsilon$. Because $d_1(A,B) = 0$, we can find $a \in A$ and $b \in B$ so that $d_1(a,b) < \delta$. This says that $d_2(f(a), f(b)) < \epsilon$, and so $d_2(f(A), f(B)) < \epsilon$. Because $\epsilon$ is arbitrary, we can conclude that $d_2(f(A), f(B)) = 0$.
(c) Consider the function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $f(x, y) = xy$. Let $A = \{(n, \frac{1}{n}) : n \in \mathbb{Z}, n > 0\}$. Let $B = \{(n, 0) : n \in \mathbb{Z}, n > 0\}$. Then $d_1(A,B) = 0$. However, $f(A) = \{1\}$, while $f(B) = \{0\}$, so $d_2(f(A), f(B)) = 1$.

5. (12 points) Show that

   $$\int_{1}^{\infty} \sin(x^2) \, dx$$
can be defined as an improper Riemann integral but not as a Lebesgue integral. *Hint:* Let \( t = x^2 \), and imitate the proof that \( \int_{0}^{\infty} \frac{\sin x}{x} \, dx \) can be defined as an improper Riemann integral but not a Lebesgue integral.

**Answer:** Let \( t = x^2 \), so that \( x = \sqrt{t} \), \( dt = 2x \, dx \), and \( dx = dt/(2\sqrt{t}) \). If \( 1 \leq x \leq \infty \), then \( 1 \leq t \leq \infty \). Therefore, as a Riemann integral, we have

\[
\int_{1}^{\infty} \sin(x^2) \, dx = \lim_{b \to \infty} \int_{1}^{b} \sin(x^2) \, dx = \lim_{b \to \infty} \int_{1}^{b} \frac{\sin t}{2\sqrt{t}} \, dt
\]

Set \( u = t^{-1/2} \), \( dv = \sin t \, dt \), \( du = -\frac{1}{2}t^{-3/2} \, dt \), and \( v = -\cos t \). We have

\[
\lim_{b \to \infty} \int_{1}^{b} \frac{\sin t}{\sqrt{t}} \, dt = \lim_{b \to \infty} \left[-\frac{\cos t}{\sqrt{t}}\right]_{1}^{b} - \frac{1}{2} \int_{1}^{b} \frac{\cos t}{t^{3/2}} \, dt.
\]

The first limit is \( \cos 1 \), and the integral converges, because \( \int_{1}^{\infty} t^{-3/2} \, dt \) converges. Therefore, the improper Riemann integral is defined.

On the other hand, if the integral converged as a Lebesgue integral, so would \( \int_{1}^{\infty} |\sin(x^2)| \, dx \). The same substitutions lead us to consider \( \lim_{b \to \infty} \int_{1}^{b} \frac{1}{\sqrt{t}} |\sin t| \, dt \). But

\[
\int_{n\pi}^{(n+1)\pi} \frac{|\sin t|}{\sqrt{t}} \, dt \geq \frac{1}{\sqrt{(n+1)\pi}} \int_{n\pi}^{(n+1)\pi} |\sin t| \, dt = \frac{1}{\sqrt{(n+1)\pi}}.
\]

Because \( \sum_{k=1}^{\infty} \frac{1}{\sqrt{(n+1)\pi}} \) diverges, we know that the Lebesgue integral must not exist.

6. (5 points) As usual, we define

\[
\frac{te^{tx}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) t^n
\]

and \( B_n = B_n(0) \). (Recall that \( B_0(x) = 1 \), \( B_1(x) = x - \frac{x}{2} \), and \( B_2(x) = x^2 - x + \frac{1}{6} \).) Show that \( B_n(\frac{1}{2}) = (2^{1-n} - 1)B_n \).

**Answer:** We have

\[
\sum_{n=0}^{\infty} \frac{B_n(\frac{1}{2})}{n!} t^n = \frac{te^{t/2}}{e^t - 1}
\]

\[
\frac{t}{e^t - 1} = 2 \cdot \frac{t/2}{e^{t/2} - 1} = 2 \sum_{n=0}^{\infty} \frac{B_n}{n!} \left(\frac{t}{2}\right)^n = \sum_{n=0}^{\infty} 2^{1-n}B_n t^n
\]

Equating coefficients, we see that \( B_n(\frac{1}{2}) + B_n = 2^{1-n}B_n \).

7. (10 points) Suppose that \( f \in C^\infty \). Let \( m \geq 1 \) be an integer. Prove using induction on \( m \) that

\[
f(0) = \int_{0}^{1} f(x) \, dx + \sum_{k=1}^{m} \frac{B_k}{k!} \left( f^{(k-1)}(1) - f^{(k-1)}(0) \right) - (-1)^m \int_{0}^{1} \frac{B_m(x)}{m!} f^{(m)}(x) \, dx
\]
Answer: We first have the case \( m = 1 \). We must show that
\[
 f(0) = \int_0^1 f(x) \, dx + \frac{B_1}{1!} (f(1) - f(0)) - (-1)^m \int_0^1 x - \frac{m}{2} f'(x) \, dx
\]
\[
= \int_0^1 f(x) \, dx - \frac{1}{2} (f(1) - f(0)) + \int_0^1 \left( x - \frac{1}{2} \right) f'(x) \, dx
\]
To check this, we integrate by parts, setting \( u = x - \frac{1}{2} \), \( du = dx \), \( dv = f'(x) \, dx \), and \( v = f(x) \). We have
\[
\int_0^1 f(x) \, dx - \frac{1}{2} (f(1) - f(0)) + \int_0^1 \left( x - \frac{1}{2} \right) f'(x) \, dx
\]
\[
= \int_0^1 f(x) \, dx - \frac{1}{2} (f(1) - f(0)) + \left( x - \frac{1}{2} \right) f(x) \bigg|_0^1 - \int_0^1 f(x) \, dx
\]
\[
= -\frac{1}{2} (f(1) - f(0)) + \frac{1}{2} f(1) + \frac{1}{2} f(0) = f(0).
\]
Now, for the inductive step, we must prove that
\[
\sum_{k=1}^m \frac{B_k}{k!} (f^{(k-1)}(1) - f^{(k-1)}(0)) - (-1)^m \int_0^1 \frac{B_m(x)}{m!} f^{(m)}(x) \, dx
\]
\[
= \sum_{k=1}^{m+1} \frac{B_k}{k!} (f^{(k-1)}(1) - f^{(k-1)}(0)) - (-1)^{m+1} \int_0^1 \frac{B_{m+1}(x)}{(m+1)!} f^{(m+1)}(x) \, dx
\]
or
\[
-(-1)^m \int_0^1 \frac{B_m(x)}{m!} f^{(m)}(x) \, dx = \frac{B_{m+1}}{(m+1)!} \left( f^{(m)}(1) - f^{(m)}(0) \right) - (-1)^{m+1} \int_0^1 \frac{B_{m+1}(x)}{(m+1)!} f^{(m+1)}(x) \, dx
\]
Multiply through by \(-1^{m+1}\), and this is the same as
\[
\int_0^1 \frac{B_m(x)}{m!} f^{(m)}(x) \, dx = (-1)^{m+1} \frac{B_{m+1}}{(m+1)!} \left( f^{(m)}(1) - f^{(m)}(0) \right) - \int_0^1 \frac{B_{m+1}(x)}{(m+1)!} f^{(m+1)}(x) \, dx
\]
We can prove this by using integration by part on the left-hand side. Let \( u = f^{(m)}(x) \), \( du = f^{(m+1)}(x) \, dx \), \( dv = \frac{B_m(x)}{m!} \, dx \), and \( v = \frac{B_{m+1}(x)}{(m+1)!} \). We have
\[
\int_0^1 \frac{B_m(x)}{m!} f^{(m)}(x) \, dx = \frac{B_{m+1}(x)}{(m+1)!} f^{(m)}(x) \bigg|_0^1 - \int_0^1 \frac{B_{m+1}(x)}{(m+1)!} f^{(m+1)}(x) \, dx
\]
\[
= \frac{1}{(m+1)!} \left( B_{m+1}(1) f^{(m)}(1) - B_{m+1}(0) f^{(m)}(0) \right) - \int_0^1 \frac{B_{m+1}(x)}{(m+1)!} f^{(m+1)}(x) \, dx
\]
\[
= \frac{B_{m+1}}{(m+1)!} \left( f^{(m)}(1) - f^{(m)}(0) \right) - \int_0^1 \frac{B_{m+1}(x)}{(m+1)!} f^{(m+1)}(x) \, dx
\]
So it remains to show that \(-1)^{m+1} B_{m+1} = B_{m+1}\). If \( m+1 \) is odd, this is true because \( B_{m+1} = 0 \). If \( m+1 \) is even, this is a trivially true inequality.

8. (28 points) Here is yet another way to compute \( \int_0^\infty \frac{\sin x}{x} \, dx \). Recall the following facts, some of which might be needed to do this problem:
\[
D_n(x) := \sum_{k=-n}^n e^{ikx}
\]
\[
D_n(x) = \frac{\sin(n + \frac{1}{2})}{\sin(\frac{x}{2})}
\]
\[
K_n(x) := \frac{1}{n+1} \sum_{k=0}^n D_k(x)
\]
\[ K_n(x) = \left( \frac{1}{n+1} \right) \frac{1 - \cos(n+1)x}{1 - \cos x} = \left( \frac{1}{n+1} \right) \left( \frac{\sin(n+1)x/2}{\sin(x/2)} \right)^2 \]

\[ \frac{1}{\pi} \int_0^\pi D_n(x) \, dx = \frac{1}{\pi} \int_0^\pi K_n(x) \, dx = 1 \]

\[ 1 - \cos x = 2 \sin^2(x/2) \]

(a) Suppose that \( g \) is Riemann-integrable on \([a, b]\) with \(|g| \leq M\). Show that

\[ \left| \int_a^b g(x) \left( \frac{1 - \cos kx}{k} \right) \, dx \right| \leq \frac{2(b-a)M}{k} \]

and conclude that

\[ \lim_{k \to \infty} \int_a^b g(x) \left( \frac{1 - \cos kx}{k} \right) \, dx = 0. \]

(b) Show that

\[ \int_0^\pi \left( \frac{1}{k} \right) \left( \frac{1 - \cos kx}{4 \sin^2 \frac{x}{2}} \right) \, dx = \frac{\pi}{2} \]

(c) Let

\[ f(x) = \frac{1}{x^2} - \frac{1}{4 \sin^2 \frac{x}{2}}, \quad 0 < x \leq \pi \]

Show that it is possible to define \( f(0) \) so that \( f(x) \) is continuous and bounded on \([0, \pi]\).

(d) Show that

\[ \lim_{k \to \infty} \int_0^\pi f(x) \left( \frac{1 - \cos kx}{k} \right) \, dx = 0. \]

(e) Show that

\[ \lim_{k \to \infty} \int_0^\pi \frac{1}{x^2} \left( \frac{1 - \cos kx}{k} \right) \, dx = \frac{\pi}{2} \]

and then

\[ \lim_{k \to \infty} \int_0^\pi \frac{\sin^2(kx/2)}{kx^2} \, dx = \frac{\pi}{4}. \]

(f) Let \( y = \frac{kx}{2} \) and conclude that

\[ \frac{\pi}{2} = \lim_{k \to \infty} \int_0^{k\pi/2} \frac{\sin^2 x}{x^2} \, dx = \int_0^{\infty} \frac{\sin^2 x}{x^2} \, dx. \]

(g) Finally, integrate by parts to show that

\[ \int_0^\infty \frac{\sin^2 x}{x^2} \, dx = \int_0^\infty \frac{\sin x}{x} \, dx. \]


(a) This is trivial:

\[ \left| \int_a^b g(x) \left( \frac{1 - \cos kx}{k} \right) \, dx \right| \leq \int_a^b \left| g(x) \left( \frac{1 - \cos kx}{k} \right) \right| \, dx \leq \int_a^b \frac{M}{k} 2 \, dx \leq \frac{2M(b-a)}{k}. \]
The limit follows immediately.

(b) We have

\[
\frac{1}{\pi} \int_0^\pi K_{k-1}(x) \, dx = 1
\]

\[
\int_0^\pi K_{k-1}(x) \, dx = \pi
\]

\[
\int_0^\pi \left( \frac{1}{k} \right) \left( \frac{1 - \cos kx}{1 - \cos x} \right) \, dx = \pi
\]

\[
\int_0^\pi \left( \frac{1}{k} \right) \left( \frac{1 - \cos kx}{2\sin^2(x/2)} \right) \, dx = \pi
\]

\[
\left( \frac{1}{k} \right) \left( \frac{1 - \cos kx}{4\sin^4(x/2)} \right) \, dx = \frac{\pi}{2}
\]

(c) Recall that \( \sin y = y - \frac{y^3}{6} + \cdots \). This means that \( \sin(x/2) = \frac{x}{2} - \frac{x^3}{48} + \cdots \) and \( \sin^2(x/2) = \frac{x^2}{4} - \frac{x^4}{48} + \cdots \).

Therefore,

\[
f(x) = \frac{1}{x^2} - \frac{1}{4\sin^2 \frac{x}{2}} = \frac{4\sin^2 \frac{x}{2} - x^2}{4x^2\sin^2 \frac{x}{2}} = \frac{(x^2 - \frac{x^2}{12} + \cdots) - x^2}{x^4 - \cdots} = -\frac{x^4/12 + \cdots}{x^4 + \cdots} = -\frac{1/12 + \cdots}{1 + \cdots}
\]

Therefore, the function is actually analytic at \( x = 0 \), if we define \( f(0) \) to be \( -\frac{1}{12} \).

(d) Because \( f(x) \) satisfies the hypotheses of part (a), this follows immediately from (a).

(e) We have

\[
\lim_{k \to \infty} \int_0^\pi \left( \frac{1}{x^2} - \frac{1}{4\sin^2 \frac{x}{2}} \right) \left( \frac{1 - \cos kx}{k} \right) \, dx = 0
\]

\[
\lim_{k \to \infty} \int_0^\pi \left( \frac{1}{x^2} \right) \left( \frac{1 - \cos kx}{k} \right) \, dx - \int_0^\pi \left( \frac{1}{4\sin^2 \frac{x}{2}} \right) \left( \frac{1 - \cos kx}{k} \right) \, dx = 0
\]

\[
\lim_{k \to \infty} \int_0^\pi \left( \frac{1}{x^2} \right) \left( \frac{1 - \cos kx}{k} \right) \, dx = \frac{\pi}{2}
\]

\[
\lim_{k \to \infty} \int_0^\pi \frac{2\sin^2 \frac{kx}{2}}{kx^2} \, dx = \frac{\pi}{2}
\]

\[
\lim_{k \to \infty} \int_0^\pi \frac{\sin^2 \frac{kx}{2}}{kx^2} \, dx = \frac{\pi}{4}
\]

(f) Let \( y = \frac{kx}{2} \), so \( x = \frac{2y}{k} \), and then \( dx = \frac{2}{k} dy \). Note that if \( 0 \leq x \leq \pi \), then \( 0 \leq y \leq \frac{k\pi}{2} \). We have

\[
\frac{\pi}{4} = \lim_{k \to \infty} \int_0^\pi \frac{\sin^2 \frac{kx}{2}}{kx^2} \, dx = \lim_{k \to \infty} \int_0^{\frac{k\pi}{2}} \frac{\sin^2 \frac{kx}{2}}{kx^2} \left( \frac{2}{k} \right) \, dy = 2 \lim_{k \to \infty} \int_0^{\frac{k\pi}{2}} \frac{\sin^2 \frac{kx}{2}}{kx^2} \, dy
\]

\[
= \frac{1}{2} \lim_{k \to \infty} \int_0^{\frac{k\pi}{2}} \frac{\sin^2 \frac{kx}{2}}{(k^2x^2/2^2)} \, dy = \frac{1}{2} \lim_{k \to \infty} \int_0^{\frac{k\pi}{2}} \frac{\sin^2 y}{y^2} \, dy = \frac{1}{2} \int_0^{\infty} \frac{\sin^2 y}{y^2} \, dy
\]

\[
\frac{\pi}{2} = \int_0^{\infty} \frac{\sin^2 y}{y^2} \, dy
\]

(g) Integration by parts is not quite trivial. We set \( u = \sin^2 x \), \( dv = x^{-2} \, dx \), \( du = 2\sin x \cos x \, dx = \sin(2x) \, dx \), and \( v = -\frac{1}{x} \). We have

\[
\int_0^{\infty} \frac{\sin^2 x}{x^2} \, dx = -\left[ \frac{\sin^2 x}{x} \right]_0^{\infty} + \int_0^{\infty} \frac{\sin(2x)}{x} \, dx
\]
Now, we can compute that $\lim_{x \to 0} \frac{\sin^2 x}{x} = 0$, and $\lim_{x \to \infty} \frac{\sin^2 x}{x} = 0$ as well. For the rest, we substitute $y = 2x$, and then $dy = 2 \, dx$, so $dx = \frac{1}{2} \, dy$:

$$\int_0^\infty \frac{\sin(2x)}{x} \, dx = \int_0^\infty \frac{\sin(2x)}{2x} \, dy = \int_0^\infty \frac{\sin y}{y} \, dy$$