Mathematics 805
Final Examination
Answers

1. *(5 points)* State the Weierstrass M-test.

Answer: Suppose that $A \subset \mathbf{R}$, and $f_n : A \to \mathbf{R}$. Suppose further that $|f_n(x)| \leq M_n$ for all $x \in A$, and that $\sum M_n$ converges. Then $\sum f_n(x)$ converges uniformly on A.

2. *(15 points)* (a) State Lebesgue's Dominated Convergence Theorem.

(b) State Lebesgue's Monotone Convergence Theorem.

(c) Define

$$f_n(x) = \begin{cases} \dfrac{1}{n} & x \in [0, n] \\ 0 & \text{otherwise} \end{cases}$$

Note that $\displaystyle\int f_n(x)\,dx = 1$, but $\displaystyle\lim_{n\to\infty} f_n(x) = 0$. Why does this contradict neither the Dominated nor the Monotone Convergence Theorems?

Answer: (a) Suppose that $f_n \in L^1$ and $\lim f_n = f$ almost everyone. Suppose further that $|f_n| \leq g$ almost everywhere, with $g \in L^1$. Then $f \in L^1$ and $\int f = \lim \int f_n$.

(b) Suppose that $f_n \in L^1$ is a monotone sequence, and suppose further that $\int f_n$ is bounded. Then f_n converges almost everywhere to a function $f \in L^1$, and $\int f = \lim \int f_n$.

(c) This example does not violate the Dominated Convergence Theorem, because there is no function $g \in L^1$ with $|f_n| \leq g$. The example does not violate the Monotone Convergence Theorem because the sequence f_n is not monotone.

3. *(10 points)* Suppose that K is a compact metric space, and $g : K \to \mathbf{R}$ a continuous function, with $g(x) > 0$ for all $x \in K$. Suppose further that $g_n : K \to \mathbf{R}$ is a sequence of continuous functions converging uniformly to g on K. Show that there is some integer N so that if $n > N$, then $g_n(x) > 0$ for all $x \in K$.

Answer: Because K is compact, we know that g attains its minimum value at some $x \in K$. In particular, we can find some ϵ so that $g(x) \geq \epsilon$ for all $x \in K$.

Because $g_n \to g$ uniformly, we can find an integer N so that if $n \geq N$, then $|g(x) - g_n(x)| \leq \epsilon/2$ for all $x \in K$. This implies that $g_n(x) \geq \epsilon/2$ for all $x \in K$, so $g_n(x) > 0$.

4. *(15 points)* (a) Let (M_1, d_1) and (M_2, d_2) be metric spaces, and $f : M_1 \to M_2$ a function. Define what is meant by "f is uniformly continuous."

(b) Suppose that (M_1, d_1) and (M_2, d_2) are metric spaces, and $f : M_1 \to M_2$ is uniformly continuous. Suppose that $A, B \subset M_1$ with $d_1(A, B) = 0$. Show that $d_2(f(A), f(B)) = 0$.

(c) Give an example to show that it is possible for $f : M_1 \to M_2$ to be continuous, with $d_1(A, B) = 0$ and $d_2(f(A), f(B)) \neq 0$.

Answer: (a) Given any $\epsilon > 0$, there is a $\delta > 0$ so that if $d_1(x, y) < \delta$, then $d_2(f(x), f(y)) < \epsilon$.

(b) Given any $\epsilon > 0$, we can find $\delta > 0$ so that if $d_1(x, y) < \delta$, then $d_2(f(x), f(y)) < \epsilon$. Because $d_1(A, B) = 0$, we can find $a \in A$ and $b \in B$ so that $d_1(a, b) < \delta$. This says that $d_2(f(a), f(b)) < \epsilon$, and so $d_2(f(A), f(B)) < \epsilon$. Because ϵ is arbitrary, we can conclude that $d_2(f(A), f(B)) = 0$.

(c) Consider the function $f : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ defined by $f(x, y) = xy$. Let $A = \{(n, \frac{1}{n}) : n \in \mathbf{Z}, n > 0\}$. Let $B = \{(n, 0) : n \in \mathbf{Z}, n > 0\}$. Then $d_1(A, B) = 0$. However, $f(A) = \{1\}$, while $f(B) = \{0\}$, so $d_2(f(A), f(B)) = 1$.

5. *(12 points)* Show that

$$\int_1^\infty \sin(x^2)\,dx$$

can be defined as an improper Riemann integral but not as a Lebesgue integral. *Hint:* Let $t = x^2$, and imitate the proof that $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx$ can be defined as an improper Riemann integral but not a Lebesgue integral.

Answer: Let $t = x^2$, so that $x = \sqrt{t}$, $dt = 2x\,dx$, and $dx = dt/(2\sqrt{t})$. If $1 \le x \le \infty$, then $1 \le t \le \infty$. Therefore, as a Riemann integral, we have

$$\int_1^\infty \sin(x^2)\,dx = \lim_{b\to\infty}\int_1^b \sin(x^2)\,dx = \lim_{b\to\infty}\int_1^b \frac{\sin t}{2\sqrt{t}}\,dt$$

Set $u = t^{-1/2}$, $dv = \sin t\,dt$, $du = -\frac{1}{2}t^{-3/2}\,dt$, and $v = -\cos t$. We have

$$\lim_{b\to\infty}\int_1^b \frac{\sin t}{\sqrt{t}}\,dt = \lim_{b\to\infty} -\frac{\cos t}{\sqrt{t}}\Big]_1^b - \frac{1}{2}\int_1^b \frac{\cos t}{t^{3/2}}\,dt.$$

The first limit is $\cos 1$, and the integral converges, because $\displaystyle\int_1^\infty t^{-3/2}\,dt$ converges. Therefore, the improper Riemann integral is defined.

On the other hand, if the integral converged as a Lebesgue integral, so would $\displaystyle\int_1^\infty |\sin(x^2)|\,dx$. The same substitutions lead us to consider $\displaystyle\lim_{b\to\infty}\int_1^b \frac{|\sin t|}{\sqrt{t}}\,dt$. But

$$\int_{n\pi}^{(n+1)\pi} \frac{|\sin t|}{\sqrt{t}}\,dt \ge \frac{1}{\sqrt{(n+1)\pi}}\int_{n\pi}^{(n+1)\pi} |\sin t|\,dt = \frac{1}{\sqrt{(n+1)\pi}}.$$

Because $\displaystyle\sum \frac{1}{\sqrt{(n+1)\pi}}$ diverges, we know that the Lebesgue integral must not exist.

6. *(5 points)* As usual, we define

$$\frac{te^{tx}}{e^t-1} = \sum_{n=0}^\infty \frac{B_n(x)}{n!}t^n$$

and $B_n = B_n(0)$. (Recall that $B_0(x) = 1$, $B_1(x) = x - \frac{1}{2}$, and $B_2(x) = x^2 - x + \frac{1}{6}$.) Show that $B_n(\frac{1}{2}) = (2^{1-n} - 1)B_n$.

Answer: We have

$$\begin{aligned}
\sum_{n=0}^\infty \frac{B_n(\frac{1}{2})}{n!}t^n &= \frac{te^{\frac{t}{2}}}{e^t-1}\\
\sum_{n=0}^\infty \frac{B_n}{n!}t^n &= \frac{t}{e^t-1}\\
\sum_{n=0}^\infty \frac{B_n(\frac{1}{2})+B_n}{n!}t^n &= \frac{te^{\frac{t}{2}}}{e^t-1} + \frac{t}{e^t-1} = \frac{t(e^{\frac{t}{2}}+1)}{e^t-1} = \frac{t(e^{\frac{t}{2}}+1)}{(e^{\frac{t}{2}}-1)(e^{\frac{t}{2}}+1)}\\
&= \frac{t}{e^{\frac{t}{2}}-1} = 2\frac{t/2}{e^{\frac{t}{2}}-1} = 2\sum_{n=0}^\infty \frac{B_n}{n!}\left(\frac{t}{2}\right)^n = \sum_{n=0}^\infty \frac{2^{1-n}B_n}{n!}t^n
\end{aligned}$$

Equating coefficients, we see that $B_n(\frac{1}{2}) + B_n = 2^{1-n}B_n$.

7. *(10 points)* Suppose that $f \in C^\infty$. Let $m \ge 1$ be an integer. Prove using induction on m that

$$f(0) = \int_0^1 f(x)\,dx + \sum_{k=1}^m \frac{B_k}{k!}\left(f^{(k-1)}(1) - f^{(k-1)}(0)\right) - (-1)^m\int_0^1 \frac{B_m(x)}{m!}f^{(m)}(x)\,dx$$

Answer: We first have the case $m = 1$. We must show that

$$\begin{aligned} f(0) &= \int_0^1 f(x)\,dx + \frac{B_1}{1!}(f(1) - f(0)) - (-1)\int_0^1 \frac{x - \frac{1}{2}}{1} f'(x)\,dx \\ &= \int_0^1 f(x)\,dx - \frac{1}{2}(f(1) - f(0)) + \int_0^1 \left(x - \frac{1}{2}\right) f'(x)\,dx \end{aligned}$$

To check this, we integrate by parts, setting $u = x - \frac{1}{2}$, $du = dx$, $dv = f'(x)\,dx$, and $v = f(x)$. We have

$$\begin{aligned} &\int_0^1 f(x)\,dx - \frac{1}{2}(f(1) - f(0)) + \int_0^1 \left(x - \frac{1}{2}\right) f'(x)\,dx \\ &= \int_0^1 f(x)\,dx - \frac{1}{2}(f(1) - f(0)) + \left(x - \frac{1}{2}\right) f(x)\Big]_0^1 - \int_0^1 f(x)\,dx \\ &= -\frac{1}{2}(f(1) - f(0)) + \frac{1}{2}f(1) + \frac{1}{2}f(0) = f(0). \end{aligned}$$

Now, for the inductive step, we must prove that

$$\begin{aligned} &\sum_{k=1}^{m} \frac{B_k}{k!}\left(f^{(k-1)}(1) - f^{(k-1)}(0)\right) - (-1)^m \int_0^1 \frac{B_m(x)}{m!} f^{(m)}(x)\,dx \\ &= \sum_{k=1}^{m+1} \frac{B_k}{k!}\left(f^{(k-1)}(1) - f^{(k-1)}(0)\right) - (-1)^{m+1} \int_0^1 \frac{B_{m+1}(x)}{(m+1)!} f^{(m+1)}(x)\,dx \end{aligned}$$

or

$$-(-1)^m \int_0^1 \frac{B_m(x)}{m!} f^{(m)}(x)\,dx = \frac{B_{m+1}}{(m+1)!}\left(f^{(m)}(1) - f^{(m)}(0)\right) - (-1)^{m+1} \int_0^1 \frac{B_{m+1}(x)}{(m+1)!} f^{(m+1)}(x)\,dx$$

Multiply through by $(-1)^{m+1}$, and this is the same as

$$\int_0^1 \frac{B_m(x)}{m!} f^{(m)}(x)\,dx = (-1)^{m+1}\frac{B_{m+1}}{(m+1)!}\left(f^{(m)}(1) - f^{(m)}(0)\right) - \int_0^1 \frac{B_{m+1}(x)}{(m+1)!} f^{(m+1)}(x)\,dx$$

We can prove this by using integration by part on the left-hand side. Let $u = f^{(m)}(x)$, $du = f^{(m+1)}(x)\,dx$, $dv = \frac{B_m(x)}{m!}\,dx$, and $v = \frac{B_{m+1}(x)}{(m+1)!}$. We have

$$\begin{aligned} \int_0^1 \frac{B_m(x)}{m!} f^{(m)}(x)\,dx &= \frac{B_{m+1}(x)}{(m+1)!} f^{(m)}(x)\Big]_0^1 - \int_0^1 \frac{B_{m+1}(x)}{(m+1)!} f^{(m+1)}(x)\,dx \\ &= \frac{1}{(m+1)!}\left(B_{m+1}(1)f^{(m)}(1) - B_{m+1}(0)f^{(m)}(0)\right) - \int_0^1 \frac{B_{m+1}(x)}{(m+1)!} f^{(m+1)}(x)\,dx \\ &= \frac{B_{m+1}}{(m+1)!}\left(f^{(m)}(1) - f^{(m)}(0)\right) - \int_0^1 \frac{B_{m+1}(x)}{(m+1)!} f^{(m+1)}(x)\,dx \end{aligned}$$

So it remains to show that $(-1)^{m+1}B_{m+1} = B_{m+1}$. If $m+1$ is odd, this is true because $B_{m+1} = 0$. If $m+1$ is even, this is a trivially true inequality.

8. *(28 points)* Here is yet another way to compute $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx$. Recall the following facts, some of which might be needed to do this problem:

$$\begin{aligned} D_n(x) &:= \sum_{k=-n}^{n} e^{ikx} \\ D_n(x) &= \frac{\sin(n + \frac{1}{2})x}{\sin(\frac{x}{2})} \\ K_n(x) &:= \frac{1}{n+1}\sum_{k=0}^{n} D_k(x) \end{aligned}$$

$$K_n(x) = \left(\frac{1}{n+1}\right)\left(\frac{1-\cos(n+1)x}{1-\cos x}\right)$$
$$= \left(\frac{1}{n+1}\right)\left(\frac{\sin(n+1)x/2}{\sin(x/2)}\right)^2$$
$$\frac{1}{\pi}\int_0^\pi D_n(x)\,dx = \frac{1}{\pi}\int_0^\pi K_n(x)\,dx = 1$$
$$1-\cos x = 2\sin^2(x/2)$$

(a) Suppose that g is Riemann-integrable on $[a,b]$ with $|g| \le M$. Show that

$$\left|\int_a^b g(x)\left(\frac{1-\cos kx}{k}\right)dx\right| \le \frac{2(b-a)M}{k}$$

and conclude that

$$\lim_{k\to\infty}\int_a^b g(x)\left(\frac{1-\cos kx}{k}\right)dx = 0.$$

(b) Show that

$$\int_0^\pi \left(\frac{1}{k}\right)\left(\frac{1-\cos kx}{4\sin^2\frac{x}{2}}\right)dx = \frac{\pi}{2}$$

(c) Let

$$f(x) = \frac{1}{x^2} - \frac{1}{4\sin^2\frac{x}{2}}, \qquad 0 < x \le \pi$$

Show that it is possible to define $f(0)$ so that $f(x)$ is continuous and bounded on $[0,\pi]$.

(d) Show that

$$\lim_{k\to\infty}\int_0^\pi f(x)\left(\frac{1-\cos kx}{k}\right)dx = 0.$$

(e) Show that

$$\lim_{k\to\infty}\int_0^\pi \frac{1}{x^2}\left(\frac{1-\cos kx}{k}\right)dx = \frac{\pi}{2}$$

and then

$$\lim_{k\to\infty}\int_0^\pi \frac{\sin^2(kx/2)}{kx^2}\,dx = \frac{\pi}{4}.$$

(f) Let $y = \frac{kx}{2}$ and conclude that

$$\frac{\pi}{2} = \lim_{k\to\infty}\int_0^{k\pi/2}\frac{\sin^2 x}{x^2}\,dx = \int_0^\infty \frac{\sin^2 x}{x^2}\,dx.$$

(g) Finally, integrate by parts to show that

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \int_0^\infty \frac{\sin x}{x}\,dx.$$

Answer: This problem is taken from "An elementary method for evaluating in infinite integral," by M.R. Spiegel, *The American Mathematical Monthly*, **58**:8, Oct. 1951, pp. 555–558.

(a) This is trivial:

$$\left|\int_a^b g(x)\left(\frac{1-\cos kx}{k}\right)dx\right| \le \int_a^b \left|g(x)\left(\frac{1-\cos kx}{k}\right)\right| dx \le \int_a^b \frac{M}{k}2\,dx \le \frac{2M(b-a)}{k}.$$

The limit follows immediately.

(b) We have

$$\frac{1}{\pi}\int_0^\pi K_{k-1}(x)\,dx = 1$$

$$\int_0^\pi K_{k-1}(x)\,dx = \pi$$

$$\int_0^\pi \left(\frac{1}{k}\right)\left(\frac{1-\cos kx}{1-\cos x}\right)\,dx = \pi$$

$$\int_0^\pi \left(\frac{1}{k}\right)\left(\frac{1-\cos kx}{2\sin^2(x/2)}\right)\,dx = \pi$$

$$\int_0^\pi \left(\frac{1}{k}\right)\left(\frac{1-\cos kx}{4\sin^2(x/2)}\right)\,dx = \frac{\pi}{2}$$

(c) Recall that $\sin y = y - \frac{y^3}{6} + \cdots$. This means that $\sin(x/2) = \frac{x}{2} - \frac{x^3}{48} + \cdots$ and $\sin^2(x/2) = \frac{x^2}{4} - \frac{x^4}{48} + \cdots$. Therefore,

$$f(x) = \frac{1}{x^2} - \frac{1}{4\sin^2\frac{x}{2}} = \frac{4\sin^2\frac{x}{2} - x^2}{4x^2\sin^2\frac{x}{2}} = \frac{(x^2 - \frac{x^4}{12} + \cdots) - x^2}{x^4 - \cdots} = \frac{-x^4/12 + \cdots}{x^4 + \cdots} = \frac{-1/12 + \cdots}{1 + \cdots}$$

Therefore, the function is actually analytic at $x = 0$, if we define $f(0)$ to be $-\frac{1}{12}$.

(d) Because $f(x)$ satisfies the hypotheses of part (a), this follows immediately from (a).

(e) We have

$$\lim_{k\to\infty}\int_0^\pi \left(\frac{1}{x^2} - \frac{1}{4\sin^2\frac{x}{2}}\right)\left(\frac{1-\cos kx}{k}\right)\,dx = 0$$

$$\lim_{k\to\infty}\int_0^\pi \left(\frac{1}{x^2}\right)\left(\frac{1-\cos kx}{k}\right)\,dx - \int_0^\pi \left(\frac{1}{4\sin^2\frac{x}{2}}\right)\left(\frac{1-\cos kx}{k}\right)\,dx = 0$$

$$\lim_{k\to\infty}\int_0^\pi \left(\frac{1}{x^2}\right)\left(\frac{1-\cos kx}{k}\right)\,dx - \frac{\pi}{2} = 0$$

$$\lim_{k\to\infty}\int_0^\pi \left(\frac{1}{x^2}\right)\left(\frac{1-\cos kx}{k}\right)\,dx = \frac{\pi}{2}$$

$$\lim_{k\to\infty}\int_0^\pi \frac{2\sin^2\frac{kx}{2}}{kx^2}\,dx = \frac{\pi}{2}$$

$$\lim_{k\to\infty}\int_0^\pi \frac{\sin^2\frac{kx}{2}}{kx^2}\,dx = \frac{\pi}{4}$$

(f) Let $y = \frac{kx}{2}$, so $x = \frac{2y}{k}$, and then $dx = \frac{2}{k}\,dy$. Note that if $0 \le x \le \pi$, then $0 \le y \le \frac{k\pi}{2}$. We have

$$\begin{aligned}
\frac{\pi}{4} &= \lim_{k\to\infty}\int_0^\pi \frac{\sin^2\frac{kx}{2}}{kx^2}\,dx = \lim_{k\to\infty}\int_0^{\frac{k\pi}{2}} \frac{\sin^2\frac{kx}{2}}{kx^2}\left(\frac{2}{k}\right)\,dy = 2\lim_{k\to\infty}\int_0^{\frac{k\pi}{2}} \frac{\sin^2\frac{kx}{2}}{k^2x^2}\,dy \\
&= \frac{1}{2}\lim_{k\to\infty}\int_0^{\frac{k\pi}{2}} \frac{\sin^2\frac{kx}{2}}{(k^2x^2/2^2)}\,dy = \frac{1}{2}\lim_{k\to\infty}\int_0^{\frac{k\pi}{2}} \frac{\sin^2 y}{y^2}\,dy = \frac{1}{2}\int_0^\infty \frac{\sin^2 y}{y^2}\,dy \\
\frac{\pi}{2} &= \int_0^\infty \frac{\sin^2 y}{y^2}\,dy
\end{aligned}$$

(g) Integration by parts is not quite trivial. We set $u = \sin^2 x$, $dv = x^{-2}\,dx$, $du = 2\sin x\cos x\,dx = \sin(2x)\,dx$, and $v = -x^{-1}$. We have

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = -\left.\frac{\sin^2 x}{x}\right]_0^\infty + \int_0^\infty \frac{\sin(2x)}{x}\,dx$$

Now, we can compute that $\lim_{x \to 0} \frac{\sin^2 x}{x} = 0$, and $\lim_{x \to \infty} \frac{\sin^2 x}{x} = 0$ as well. For the rest, we substitute $y = 2x$, and then $dy = 2\,dx$, so $dx = \frac{1}{2}\,dy$:

$$\int_0^\infty \frac{\sin(2x)}{x}\,dx = \int_0^\infty \frac{\sin(2x)}{2x}\,dy = \int_0^\infty \frac{\sin y}{y}\,dy$$