Mathematics 805
Homework 1
Due Friday, January 23, 1 PM

1. **5.1.1.** Let $(\mathbf{M}, d)$ be a metric space. Define the maps $d_1, d_2 : \mathbf{M} \times \mathbf{M} \to \mathbf{R}$ by $d_1 := d/(1+d)$ and $d_2 = \min(1, d)$; i.e., for any $x, y \in \mathbf{M}$, $d_1(x,y) = d(x,y)/(1+d(x,y))$ and $d_2(x,y) = \min\{1, d(x,y)\}$. Show that d_1 and d_2 are both metrics on $\mathbf{M}$ and that we have $d_1 \le d_2 \le 2d_1$.

Answer: First, we show that d_1 is a metric. Clearly, $d_1(x,y) \ge 0$. If $d_1(x,y) = 0$, then $d(x,y) = 0$, so $x = y$, and $d_1(x,x) = 0$. Clearly, $d_1(x,y) = d_1(y,x)$. It remains to show that the triangle inequality holds for d_1.

We start with a trivial observation. The function $f(t) = t/(1+t)$ is strictly increasing for $t \ge 0$. (One way to see this is to notice that $f'(t) > 0$.) Therefore, if $d(a,b) \le d(c,d)$, then $d_1(a,b) \le d_1(c,d)$.

We need to show that $d_1(x,z) \le d_1(x,y) + d_1(y,z)$. We know that $d(x,z) \le d(x,y) + d(y,z)$. We can divide by $1 + d(x,z)$, and then we have

$$\frac{d(x,z)}{1+d(x,z)} \le \frac{d(x,y)}{1+d(x,z)} + \frac{d(y,z)}{1+d(x,z)}.$$

If $d(x,z) \ge d(x,y)$ and $d(x,z) \ge d(y,z)$, then

$$\frac{d(x,y)}{1+d(x,z)} + \frac{d(y,z)}{1+d(x,z)} \le \frac{d(x,y)}{1+d(x,y)} + \frac{d(y,z)}{1+d(y,z)} = d_1(x,y) + d_1(y,z),$$

and therefore, we are done.

Suppose that $d(x,z) < d(x,y)$. In that case, $d_1(x,z) < d_1(x,y)$, and therefore $d_1(x,z) < d_1(x,y) + d_1(y,z)$.

Finally, if $d(x,z) < d(y,z)$, then $d_1(x,z) < d_1(y,z)$ and therefore $d_1(x,z) < d_1(x,y) + d_1(y,z)$. This shows that d_1 satisfies the triangle inequality and therefore is a metric.

Second, we show that d_2 is a metric. We clearly have $d_2(x,y) \ge 0$, and $d_2(x,y) = 0$ if and only if $x = y$. It is also clear that $d_2(x,y) = d_2(y,x)$. Again, the triangle inequality looms as the obstacle.

If $d(x,z) \le d(x,y)$, then $d_2(x,z) = \min\{1, d(x,z)\} \le \min\{1, d(x,y)\} = d_2(x,y) \le d_2(x,y) + d_2(y,z)$. Similarly, if $d(x,z) \le d(y,z)$, then $d_2(x,z) = \min\{1, d(x,z)\} \le \min\{1, d(y,z)\} = d_2(y,z) \le d_2(x,y) + d_2(y,z)$. We therefore need only worry about the case that $d(x,z) > d(x,y)$ and $d(x,z) > d(y,z)$.

If $d(x,z) \le 1$, then $d_2(x,z) = d(x,z)$, $d_2(x,y) = d(x,y)$ and $d_2(y,z) = d(y,z)$, so we are done. If $d(x,z) > 1$, then $d_2(x,z) = 1$. If $d(x,y) > 1$ or $d(y,z) > 1$, we are done, so we may assume that $d(x,y) < 1$ and $d(y,z) < 1$. In that case, we know that $d(x,z) < d(x,y) + d(y,z)$, which implies that $1 < d(x,y) + d(y,z) = d_2(x,y) + d_2(y,z)$. That exhausts all cases and shows that d_2 is a metric.

We need to show that $d_1 < d_2$. We know that $d_1(x,y) < 1$ and $d_1(x,y) < d(x,y)$, which shows that $d_1(x,y) < \min\{1, d(x,y)\} = d_2(x,y)$.

Finally, we need to show that $d_2 \le 2d_1$. Note that $d_1(x,y) = 1 - \frac{1}{1+d(x,y)}$. If $d(x,y) \ge 1$, then $d_2(x,y) = 1$ while $d_1(x,y) \ge \frac{1}{2}$, and therefore $d_2 \le 2d_1$. If $d(x,y) < 1$, then $d_2(x,y) = d(x,y)$ while $d_1(x,y) > d(x,y)/2$, allowing us to conclude that $d_2 \le 2d_1$.

2. **5.1.2.** Consider the set $\mathbf{R}^n = \{(x_1, x_2, \ldots x_n) : x_k \in \mathbf{R} \text{ for } 1 \le k \le n\}$. Define the maps

$$d_{\mathrm{euc}}(x,y) := \sqrt{\sum_{k=1}^{n} (x_k - y_k)^2}$$

$$d_{\max}(x,y) := \max\{|x_1 - y_1|, \cdots, |x_n - y_n|\}$$

$$d_{\mathrm{sum}}(x,y) := \sum_{k=1}^{n} |x_k - y_k|$$

Show that d_{euc}, $d_{\max}$, and d_{sum} are metrics on $\mathbf{R}^n$, and that we have the inequalities $d_{\max} \le d_{\mathrm{euc}} \le d_{\mathrm{sum}} \le n d_{\max}$.

Answer: To show that this theorem in fact is a general property of products of metric spaces, I will write $|x_k - y_k|$ as $d(x_k, y_k)$.

We easily see that $d_{\text{euc}}(x, y) \geq 0$, and that $d_{\text{euc}}(x, y) = 0$ if and only if $x = y$. The triangle inequality is not so clear.

We have

$$\begin{aligned} d_{\text{euc}}(x, z) &= \sqrt{\sum d(x_k, z_k)^2} \leq \sqrt{\sum (d(x_k, y_k) + d(y_k, z_k))^2} \\ &\leq \sqrt{\sum d(x_k, y_k)^2} + \sqrt{\sum d(y_k, z_k)^2} = d_{\text{euc}}(x, y) + d_{\text{euc}}(y, z) \end{aligned}$$

The mysterious inequality at the start of the second line is the Triangle Inequality as stated on page 41.

We clearly have $d_{\max}(x, y) \geq 0$ and $d_{\max}(x, y) = 0$ if and only if $x = y$. Furthermore, it is immediate that $d_{\max}(x, y) = d_{\max}(y, z)$.

The triangle inequality is not too bad this time:

$$\begin{aligned} d_{\max}(x, z) &= \max\{d(x_1, z_1), \cdots, d(x_n, z_n)\} \\ &\leq \max\{d(x_1, y_1) + d(y_1, z_1), \cdots, d(x_n, y_n) + d(y_n, z_n)\} \\ &\leq \max\{d(x_1, y_1), \cdots, d(x_n, y_n)\} + \max\{d(y_1, z_1), \cdots, d(y_n, z_n)\} = d_{\max}(x, y) + d_{\max}(y, z) \end{aligned}$$

Again, it's easy to see that $d_{\text{sum}}(x, y) \geq 0$ and $d_{\text{sum}}(x, y) = 0$ if and only if $x = y$. It's also trivial that $d_{\text{sum}}(x, y) = d_{\text{sum}}(y, x)$. And the triangle inequality this time is also easy:

$$d_{\text{sum}}(x, z) = \sum d(x_k, z_k) \leq \sum (d(x_k, y_k) + d(y_k, z_k)) = \sum d(x_k, y_k) + \sum d(y_k, z_k) = d_{\text{sum}}(x, y) + d_{\text{sum}}(y, z).$$

Finally, the inequalities:

$$\begin{aligned} d_{\text{euc}}(x, y) &= \sqrt{\sum d(x_k, y_k)^2} \\ &\geq \sqrt{\max d(x_k, y_k)^2} = \max\{d(x_1, y_1), \cdots, d(x_n, y_n)\} = d_{\max}(x, y) \\ d_{\text{sum}}(x, y)^2 &= \left(\sum d(x_k, y_k)\right)^2 \\ &\geq \sum d(x_k, y_k)^2 = d_{\text{euc}}(x, y)^2 \\ d_{\text{sum}}(x, y) &= \sum d(x_k, y_k) \leq \sum \max\{d(x_k, y_k)\} = n d_{\max}(x, y) \end{aligned}$$

3. **5.8.2.** Let $\mathbf{M}$ be a *nonempty* set and suppose that $d : \mathbf{M} \times \mathbf{M} \to \mathbf{R}$ satisfies the following conditions:

$$\begin{aligned} &d(x, y) = 0 \iff x = y \qquad (\forall x, y \in \mathbf{M}), \text{ and} \\ &d(x, y) \leq d(x, z) + d(y, z) \qquad (\forall x, y, z \in \mathbf{M}) \end{aligned}$$

Show that $(\mathbf{M}, d)$ is a metric space.

Answer: Apply the second equation with $y = x$, and we have $d(x, x) \leq d(x, z) + d(x, z)$. Because $d(x, x) = 0$, we see that $2d(x, z) \geq 0$, so $d(x, z) \geq 0$.

Apply the second equation with $z = x$, and we get $d(x, y) \leq d(x, x) + d(y, x)$. Because $d(x, x) = 0$, we have $d(x, y) \leq d(y, x)$. But x and y are arbitrary, so we can also deduce that $d(y, x) \leq d(x, y)$, implying that $d(x, y) = d(y, x)$.

4. **5.8.3.** (a) Let $\ell^\infty(\mathbf{N})$ denote the set of all *bounded* real sequences $x \in \mathbf{R}^{\mathbf{N}}$. For each $x, y \in \ell^\infty(\mathbf{N})$, define

$$d_\infty(x, y) := \sup\{|x_n - y_n| : n \in \mathbf{N}\}.$$

Show that $(\ell^\infty(\mathbf{N}), d_\infty)$ is a metric space.

(b) Let $\ell^1(\mathbf{N})$ denote the set of all real sequences $x \in \mathbf{R}^{\mathbf{N}}$ that are *summable*, i.e., $\sum_{n=1}^{\infty} |x_n| < \infty$. For each $x, y \in \ell^1(\mathbf{N})$, define

$$d_1(x,y) := \sum_{n=1}^{\infty} |x_n - y_n|.$$

Show that $(\ell_1(\mathbf{N}), d_1)$ is a metric space.

(c) Consider the space $\ell^2(\mathbf{N})$ of all real sequences $x \in \mathbf{R}^{\mathbf{N}}$ that are *square summable*, i.e., $\sum_{n=1}^{\infty} |x_n|^2 < \infty$. For each $x, y \in \ell^2(\mathbf{N})$, define

$$d_2(x,y) := \sqrt{\sum_{n=1}^{\infty} (x_n - y_n)^2}.$$

Show that $(\ell^2(\mathbf{N}), d_2)$ is a metric space.

Answer: (a) First, note that because $\sup |x_n - y_n| \le \sup |x_n| + \sup |y_n|$, we know that d_∞ is defined. It is clear that $d_\infty(x,y) \ge 0$ and $d_\infty(x,y) = 0$ if and only if $x = y$. Because $\sup(A_n + B_n) \le \sup A_n + \sup B_n$, we have

$$\begin{aligned} d_\infty(x,z) &= \sup |x_n - z_n| = \sup |(x_n - y_n) + (y_n - z_n)| \\ &\le \sup (|x_n - y_n| + |y_n - z_n|) \le \sup |x_n - y_n| + \sup |y_n - z_n| = d_\infty(x,y) + d_\infty(y,z). \end{aligned}$$

(b) Because $\sum |x_n - y_n| \le \sum |x_n| + \sum |y_m|$, we know that $d_1(x,y)$ is defined. Clearly, $d_1(x,y) = d_1(y,x)$, $d_1(x,y) \ge 0$, and $d_1(x,y) = 0$ if and only if $x = y$. Finally,

$$\begin{aligned} d_1(x,z) &= \sum_{n=1}^{\infty} |x_n - z_n| = \sum_{n=1}^{\infty} |(x_n - y_n) + (y_n - z_n)| \le \sum_{n=1}^{\infty} |x_n - y_n| + |y_n - z_n| \\ &= \sum_{n=1}^{\infty} |x_n - y_n| + \sum_{n=1}^{\infty} |y_n - z_n| = d_1(x,y) + d_1(y,z) \end{aligned}$$

(c) The triangle inequality says that

$$\sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} \le \sqrt{\sum_{i=1}^{n} x_i^2} + \sqrt{\sum_{i=1}^{n} y_i^2}$$

We therefore have

$$\sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} \le \sqrt{\sum_{i=1}^{\infty} x_i^2} + \sqrt{\sum_{i=1}^{\infty} y_i^2}$$

and hence

$$\sqrt{\sum_{i=1}^{\infty} (x_i - y_i)^2} \le \sqrt{\sum_{i=1}^{\infty} x_i^2} + \sqrt{\sum_{i=1}^{\infty} y_i^2}$$

Therefore, $d_2(x,y)$ is defined. Again, we immediately see that $d_2(x,y) \ge 0$, that $d_2(x,y) = 0$ if and only if $x = y$, and that $d_2(x,y) = d_2(y,x)$. For the triangle inequality for d_2, we start with the usual triangle inequality:

$$\sqrt{\sum_{i=1}^{n} (x_i - z_i)^2} \le \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} + \sqrt{\sum_{i=1}^{n} (y_i - z_i)^2}$$

First, we can let $n \to \infty$ on the right-hand side of the inequality to get

$$\sqrt{\sum_{i=1}^{n} (x_i - z_i)^2} \le \sqrt{\sum_{i=1}^{\infty} (x_i - y_i)^2} + \sqrt{\sum_{i=1}^{\infty} (y_i - z_i)^2}$$

and now letting $n \to \infty$ on the left-hand side shows that $d_2(x,z) \le d_2(x,y) + d_2(y,z)$.