

Mathematics 805
Homework 2
Due Friday, January 30, 1 PM

1. **5.8.4.** Let $\mathbf{D} := \{z \in \mathbf{C} : |z| \leq 1\} \subset \mathbf{C}$ be the *closed unit disk* and, for any $z, w \in \mathbf{D}$, define

$$d(z, w) := \begin{cases} |z - w| & \text{if } z/|z| = w/|w|, \\ |z| + |w| & \text{otherwise.} \end{cases}$$

Geometrically, if two points $z, w \in \mathbf{D} \setminus \{0\}$ are on a *radius* of the unit circle, then $d(z, w)$ is their Euclidean distance. Otherwise, $d(z, w)$ is the sum of the distances of z and w from the origin. Show that (\mathbf{D}, d) is a metric space.

Answer: It is clear that $d(z, w) = d(w, z)$, and that $d(z, w) \geq 0$. It is also clear that if $d(z, w) = 0$, then $z = w$. The triangle inequality requires a lot more work.

Suppose that $u, z, w \in \mathbf{C}$. We need to show that $d(z, w) \leq d(z, u) + d(u, w)$. There are several cases. If $z = 0$, then $d(0, w) = |w|$ and $d(0, u) = |u|$. If $u/|u| = w/|w|$, then we have to show that $|w| \leq |u| + |w - u|$. This is the triangle inequality in disguise: $|w| = |u + (w - u)| \leq |u| + |w - u|$. Otherwise, $d(u, w) = |u| + |w|$, and we have the obvious inequality $|w| \leq |u| + |u| + |w|$.

Next, suppose that $z/|z| = w/|w|$. If in addition $z/|z| = u/|u|$, then we have to show that $|z - w| \leq |z - u| + |u - w|$, which is the usual triangle inequality again. If $z/|z| \neq u/|u|$, then we have to show that $|z - w| \leq |z| + |u| + |u| + |w|$, which is true because $|z - w| \leq |z| + |w|$.

Finally, suppose that $z/|z| \neq w/|w|$. If in addition, we have $z/|z| \neq u/|u|$ and $w/|w| \neq u/|u|$, then we have to prove that $|z| + |w| \leq |z| + |u| + |u| + |w|$, which of course is true. Suppose instead that $z/|z| = u/|u|$. We then need to prove that $|z| + |w| \leq |z - u| + |u| + |w|$. This is equivalent to $|z| \leq |z - u| + |u|$, which is the triangle inequality in disguise.

2. **5.8.8.** Show that in any metric space (\mathbf{M}, d) , we have

$$|d(x, y) - d(x', y')| \leq d(x, x') + d(y, y') \quad (\forall x, x', y, y' \in \mathbf{M}).$$

Answer: We have $d(x, y) \leq d(x, x') + d(x', y) \leq d(x, x') + d(x', y') + d(y', y)$, so subtraction yields $d(x, y) - d(x', y') \leq d(x, x') + d(y, y')$. Symmetry says that $d(x', y') - d(x, y) \leq d(x', x) + d(y', y) = d(x, x') + d(y, y')$. Hence, $|d(x', y') - d(x, y)| \leq d(x', x) + d(y', y)$.

3. **5.8.9.** (a) Let \mathbf{M} be a metric space. Show that

$$A \subset M \text{ is open} \iff A \cap B^- \subset (A \cap B)^- \quad (\forall B \subset M)$$

(b) Find two *open* sets $A, B \subset \mathbf{R}$ such that the sets $A \cap B^-$, $B \cap A^-$, $(A \cap B)^-$ and $A^- \cap B^-$ are all *distinct*.

(c) Find two *intervals* $I, J \subset \mathbf{R}$ such that $I \cap J^- \not\subseteq (I \cap J)^-$.

Answer: (a) We proved in class that

$$A \text{ is open} \iff \forall B \subset M \quad (A \cap B = \emptyset \implies A \cap B^- = \emptyset)$$

First, suppose that A is *not* open. Then A^c is not closed, so $(A^c)^- \neq A^c$. Let $B = A^c$, and we see that $(A \cap B)^- = (A \cap A^c)^- = \emptyset^- = \emptyset$, while $A \cap B^- = A \cap (A^c)^- \neq \emptyset$. Therefore, $A \cap B^- \not\subseteq (A \cap B)^-$.

Second, suppose that A is open. Let B be any set. Let $C = A \cap B$ and let $D = A^c \cap B$. Notice that $C \cup D = B$ and $C \cap D = \emptyset$. Notice also that $B^- = (C \cup D)^- = C^- \cup D^-$. Because $C \subset A$, we have $A \cap C = C$. Because $A \cap D = \emptyset$, we know that $A \cap D^- = \emptyset$.

Now, $A \cap B^- = A \cap (C^- \cup D^-) = (A \cap C^-) \cup (A \cap D^-) = A \cap C^-$, while $(A \cap B)^- = (A \cap (C \cup D))^-- = ((A \cap C) \cup (A \cap D))^-- = (A \cap C)^- = C^-$. So the relation $A \cap B^- \subset (A \cap B)^-$ becomes $A \cap C^- \subset C^-$, which is obviously true.

(b) Let $A = (0, 2) \cup (3, 5)$ and let $B = (1, 4) \cup (5, 6)$. Then $A^- = [0, 2] \cup [3, 5]$ and $B^- = [1, 4] \cup [5, 6]$. We have

- (i) $A \cap B^- = [1, 2] \cup (3, 4]$.
- (ii) $A^- \cap B = (1, 2] \cup [3, 4)$.
- (iii) $(A \cap B)^- = ((1, 2) \cup (3, 4))^- = [1, 2] \cup [3, 4]$.
- (iv) $A^- \cap B^- = [1, 2] \cup [3, 4] \cup \{5\}$.

(c) Let $I = [0, 1]$ and $J = (1, 2)$, so $J^- = [1, 2]$. Then $I \cup J^- = [0, 1] \cap [1, 2] = \{1\}$, and $(I \cap J)^- = \emptyset^- = \emptyset$, so indeed $I \cap J^- \not\subseteq (I \cap J)^-$.

4. **5.8.10.** Let (\mathbf{M}, d) be a metric space. For each $A \subset M$, the *derived set* of A , denoted by A' , is the set of all *limit points* of A . For any $A, B \subset M$, prove the following:

- (a) A' is *closed*.
- (b) $(A \cup B)' = A' \cup B'$.
- (c) $(A \cap B)' \subset A' \cap B'$.
- (d) $A' \setminus B' \subset (A \setminus B)'$.
- (e) If $A \subset B$ and $B \setminus A$ is *finite*, then $A' = B'$.

Answer: (a) Suppose that $x \notin A'$. We need to find an open set U so that $x \in U$ and $U \subset (A')^c$. Suppose instead that for every $\epsilon > 0$, $B_\epsilon(x) \cap A' \neq \emptyset$. For each natural number k , pick $x_k \in A' \cap B_{1/2k}(x)$. Because each $x_k \in A'$, we can find $y_k \in A$ so that $d(x_k, y_k) < \frac{1}{2k}$. Then $d(x, y_k) < \frac{1}{k}$. This means that for every $\epsilon > 0$, $B_\epsilon(x) \cap A \neq \emptyset$, meaning that $x \in A'$. This is a contradiction.

(b) If $C \subset D$, then $C' \subset D'$. We have $A \subset A \cup B$, so $A' \subset (A \cup B)'$. Similarly, $B' \subset (A \cup B)'$, and therefore $A' \cup B' \subset (A \cup B)'$.

To show the reverse inclusion, suppose that $x \in (A \cup B)'$ and $x \notin A' \cup B'$. Because $x \notin A'$, there is some $\epsilon_1 > 0$ so that $\dot{B}_{\epsilon_1}(x) \cap A = \emptyset$. Because $x \notin B'$, there is some $\epsilon_2 > 0$ so that $\dot{B}_{\epsilon_2}(x) \cap B = \emptyset$. Suppose $\epsilon = \min(\epsilon_1, \epsilon_2)$. Then $\dot{B}_\epsilon(x) \cap (A \cup B) = \emptyset$, so $x \notin (A \cup B)'$.

(c) We have $A \cap B \subset A$, so $(A \cap B)' \subset A'$. We also have $A \cap B \subset B$, so $(A \cap B)' \subset B'$. Therefore, $(A \cap B)' \subset A' \cap B'$.

(d) Pick $x \in (A' \setminus B')$. Because $x \notin B'$, there is some $\epsilon_1 > 0$ so that $\dot{B}_{\epsilon_1}(x) \cap B = \emptyset$. Because $x \in A'$, for every $\epsilon > 0$, $\dot{B}_\epsilon(x) \cap A \neq \emptyset$. Thus, if $\epsilon < \epsilon_1$, $\dot{B}_\epsilon(x) \cap (A \setminus B) \neq \emptyset$. Hence, $x \in (A \setminus B)'$.

(e) Write $B = A \cup C$, where C is a finite set. We know that $B' = A' \cup C'$, and because C is a finite set, we know that $C' = \emptyset$. Therefore, $B' = A'$.