Mathematics 805
Homework 3
Due Friday, February 6, 1 PM

1. **5.4.1.** Let $(\mathbf{M}, d)$ and $(\mathbf{M}', d')$ be metric spaces, and let $f : \mathbf{M} \to \mathbf{M}'$. Show that the following statements are pairwise equivalent.

(a) $f \in C(\mathbf{M}, \mathbf{M}')$.
(b) For every *closed* set $Y \subset \mathbf{M}'$, $f^{-1}(Y)$ is *closed* in $\mathbf{M}$.
(c) For every set $Y \subset \mathbf{M}'$, $f^{-1}(Y^\circ) \subset (f^{-1}(Y))^\circ$.
(d) For every set $Y \subset \mathbf{M}'$, $(f^{-1}(Y))^- \subset f^{-1}(Y^-)$.
(e) For every set $X \subset \mathbf{M}$, $f(X^-) \subset (f(X))^-$.

Answer: Note that if $Y \subset \mathbf{M}'$, then $f^{-1}(Y^c) = (f^{-1}(Y))^c$. That observation is helpful below.

(a)⇒(b): Suppose that $Y \subset \mathbf{M}'$ is closed. Then Y^c is open, so $f^{-1}(Y^c)$ is open. Therefore, $(f^{-1}(Y))^c$ is open, so $f^{-1}(Y)$ is closed.

(b)⇒(a): Suppose $V \subset \mathbf{M}'$ is open. Then V^c is closed, so by assumption $f^{-1}(V^c)$ is closed. This says that $f^{-1}(V)^c$ is closed, so $f^{-1}(V)$ is open. Therefore, $f \in C(\mathbf{M}, \mathbf{M}')$.

(a)⇒(c): Take $Y \subset \mathbf{M}'$. Then Y° is open, so $f^{-1}(Y^\circ)$ is also open. Also, $Y^\circ \subset Y$, so $f^{(-1)}(Y^\circ) \subset f^{-1}(Y)$. If V is any set, $V^\circ = \bigcup\limits_{\text{open } U \subset V} U$. In particular, if $U \subset V$ and U is open, then $U \subset V^\circ$. Therefore, $f^{-1}(Y^\circ) \subset (f^{-1}(Y))^\circ$.

(c)⇒(a): Take $V \subset \mathbf{M}'$ open. We must show that $f^{-1}(V)$ is open. Because V is open, $V = V^\circ$. Inclusion (c) says that $f^{-1}(V^\circ) \subset (f^{-1}(V))^\circ$. So we have $f^{-1}(V) \subset (f^{-1}(V))^\circ$. Obviously, $(f^{-1}(V))^\circ \subset f^{-1}(V)$. Therefore, $(f^{-1}(V))^\circ = f^{-1}(V)$. Because $(f^{-1}(V))^\circ$ is open, we can conclude that $f^{-1}(V)$ is open.

(b)⇒(d): Take $Y \subset \mathbf{M}'$. Then Y^- is closed, so $f^{-1}(Y^-)$ is closed. Because $Y \subset Y^-$, we know that $f^{-1}(Y) \subset f^{-1}(Y^-)$. Now, if V is any set, $V^- = \bigcap\limits_{\text{closed } F \supset V} F$, so in particular, if $V \subset F$ and F is closed, then $V^- \subset F$. Therefore, $(f^{-1}(Y))^- \subset f^{-1}(Y^-)$.

(d)⇒(b): Suppose that $F \subset \mathbf{M}'$ is closed. We must show that $f^{-1}(M)$ is closed. Because F is closed, we know that $F^- = F$ and therefore $f^{-1}(F^-) = f^{-1}(F)$. Relation (d) now says that $(f^{-1}(F))^- \subset f^{-1}(F)$. Clearly, $(f^{-1}(F))^- \supset f^{-1}(F)$. Therefore, $(f^{-1}(F))^- = f^{-1}(F)$, which shows that $f^{-1}(F)$ is closed.

(a)⇒(e): Let $X \subset \mathbf{M}$, and pick $y_0 \in f(X^-)$. Therefore, there is some $x_0 \in X^-$ so that $f(x_0) = y_0$. Because $x_0 \in X^-$, there is a sequence $(x_n) \subset X$ so that $\lim x_n = x_0$. Because f is continuous, we know that $\lim f(x_n) = f(x_0) = y_0$. Therefore, $y_0 \in (f(X))^-$.

(e)⇒(a): Here, we reason by contradiction. Suppose that $f \notin C(\mathbf{M}, \mathbf{M}')$. Then for some $x_0 \in \mathbf{M}$, there is some $\epsilon > 0$ so that for all $\delta > 0$, there is some $x \in \mathbf{M}$ so that $d(x, x_0) < \delta$ and $d(f(x), f(x_0)) > \epsilon$. Apply this with $\delta = 1, \frac{1}{2}, \frac{1}{3}, \ldots$ to find points x_n so that $d(x_n, x_0) < \frac{1}{n}$ and $d(f(x_n), f(x_0)) > \epsilon$. Let $X = \{x_1, x_2, x_3, \ldots\}$, so that $x_0 \in X^-$. Therefore, $y_0 \in f(X^-)$. However, because $d(f(x_n), y_0) > \epsilon$ for all n, we know that $y_0 \notin f(X)^-$. Therefore, $f(X^-) \nsubseteq (f(X))^-$.

2. **5.4.2.** Show that the maps $+ : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ and $\cdot : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ defined by $+(x, y) := x + y$ and $\cdot(x, y) := xy$ are continuous.

Answer: We use the metric $d_{\max}$ on $\mathbf{R} \times \mathbf{R}$, and use the metric $d(x, y) = |x - y|$ on $\mathbf{R}$.

(a) Given $\epsilon > 0$, choose $\delta = \frac{\epsilon}{2}$. If $d_{\max}((x, y), (x_0, y_0)) < \delta$, that means that $d(x, x_0) < \delta$ and $d(y, y_0) < \delta$. Then $d(x + y, x_0 + y_0) = |(x + y) - (x_0 + y_0)| \le |x - x_0| + |y - y_0| < 2\delta = \epsilon$.

(b) It seems simpler to do this problem in cases. If $x_0 = y_0 = 0$, and $\epsilon > 0$, let $\delta = \sqrt{\epsilon}$. If $d_{\max}((x, y), (0, 0)) < \delta$, then $|x| < \delta$ and $|y| < \delta$, and then $|xy - 0| < \delta^2 = \epsilon$.

Now suppose that $x_0 = 0$ and $y_0 \ne 0$. Given $\epsilon > 0$, let $\delta = \min(\frac{\epsilon}{2|y_0|}, |y_0|)$. If $d_{\max}((x, y), (0, y_0)) < \delta$, then $|y - y_0| < \delta \le |y_0|$, so $|y| = |(y - y_0) + y_0| \le |y - y_0| + |y_0| \le 2|y_0|$ and $|x| < \delta \le \frac{\epsilon}{2|y_0|}$, and therefore $|xy - 0 \cdot y_0| < (2|y_0|)\frac{\epsilon}{2|y_0|} = \epsilon$. A similar argument works if $x_0 \ne 0$ and $y_0 = 0$.

Finally, suppose that $x_0 \ne 0$ and $y_0 \ne 0$. In that case, given $\epsilon > 0$, let $\delta = \min(\frac{\epsilon}{4|y_0|}, \frac{\epsilon}{2|x_0|}, |y_0|)$. Suppose that $d_{\max}((x, y), (x_0, y_0)) < \delta$. As above, we have $|y| = |y - y_0 + y_0| \le |y - y_0| + |y_0| < \delta + |y_0| < 2|y_0|$. We then have $|xy - x_0 y_0| = |xy - x_0 y + x_0 y - x_0 y_0| \le |xy - x_0 y| + |x_0 y - x_0 y_0| = |y||x - x_0| + |x_0||y - y_0| < 2|y_0|\frac{\epsilon}{4|y_0|} + |x_0|\frac{\epsilon}{2|x_0|} \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

3. **5.4.3.** Let $\mathbf{M}_1$, $\mathbf{M}_2$, and $\mathbf{M}_3$ be metric spaces, and $f \in C(\mathbf{M}_1 \times \mathbf{M}_2, \mathbf{M}_3)$. Show that for each fixed $z_0 \in \mathbf{M}_3$, the set

$$K_{z_0} := \{(x, y) \in \mathbf{M}_1 \times \mathbf{M}_2 : f(x, y) = z_0\}$$

is *closed* in $\mathbf{M}_1 \times \mathbf{M}_2$. Deduce that, for a continuous function $f : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$, the *level curve*

$$K_c := \{(x, y) \in \mathbf{R} \times \mathbf{R} : f(x, y) = c\}$$

is a closed subset of the plane $\mathbf{R}^2$ for each constant $c \in \mathbf{R}$.

Answer: This is trivial. $K_{z_0} = f^{-1}(\{z_0\})$. Because $\{z_0\}$ is a closed set and f is a continuous function, K_{z_0} is closed.

4. **5.8.26.** Let $(\mathbf{M}, d)$ be a metric space and $D \subset \mathbf{M}$ a *dense* subset. Show that if every *Cauchy* sequence in D converges to an element of $\mathbf{M}$, then $(\mathbf{M}, d)$ is complete.

Answer: Let (x_n) be a Cauchy sequence in $\mathbf{M}$. We need to show that (x_n) converges to an element in $\mathbf{M}$.

Given an $\epsilon > 0$, we can find N_1 so that if $m, n > N_1$, then $d(x_n, x_m) < \frac{\epsilon}{3}$. Let $N = \max(N_1, \frac{3}{\epsilon})$, so that if $m > N$, then $\frac{1}{m} < \frac{\epsilon}{3}$.

Because $x_n \in \mathbf{M}$, and D is dense in $\mathbf{M}$, we can find an elements $y_n \in D$ so that $d(x_n, y_n) < \frac{1}{n}$ for $n = 1, 2, 3, \ldots$. We first show that (y_n) is Cauchy. Given $\epsilon > 0$, if $m, n > N$, then $d(y_m, y_n) \leq d(y_m, x_m) + d(x_m, x_n) + d(x_n, y_n) < \frac{1}{m} + \frac{\epsilon}{3} + \frac{1}{n} < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$. Therefore, (y_n) is Cauchy.

By hypothesis, we know that $y_n \to y \in \mathbf{M}$. It remains to show that $x_n \to y$. Given ϵ, find N_2 so that if $n > N_2$, then $d(y_n, y) < \frac{\epsilon}{2}$. Let $N_3 = \max(N_2, \frac{2}{\epsilon})$. If $n > N_3$, then $d(x_n, y) \leq d(x_n, y_n) + d(y_n, y) < \frac{1}{n} + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

5. **5.8.27.** Let $(\mathbf{M}, d)$ be a *complete* metric space and let $x \in \mathbf{M}^{\mathbf{N}}$. Show that, if $\sum\limits_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty$, then (x_n) is convergent.

Answer: We need only show that (x_n) is Cauchy. Because $S = \sum\limits_{n=1}^{\infty} d(x_n, x_{n+1})$ is a convergent sum of non-negative numbers, we know that the partial sums $S_N = \sum\limits_{n=1}^{N} d(x_n, x_{n+1})$ form a monotonically increasing Cauchy sequence with $S_n \to S$. Given $\epsilon > 0$, we can find an N so that if $m, n > N$ then $|S_m - S_{n-1}| < \epsilon$. Assume for convenience that $m > n$, and then $|S_m - S_{n-1}| = S_m - S_{n-1} = \sum\limits_{k=n}^{m} d(x_k, x_{k+1})$.

Finally, we can see that $d(x_m, x_n) \leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \cdots + d(x_{n+1}, x_n) = S_{m-1} - S_{n-1} \leq S_m - S_{n-1} < \epsilon$, showing that (x_m) is Cauchy.