Mathematics 805
Homework 4
Due Friday, February 13, 1 PM

1. **5.5.1.** Let $(\mathbf{M}, d)$ be a metric space, and $A, B \subset \mathbf{M}$ two disjoint nonempty closed sets. Define $f : \mathbf{M} \to \mathbf{R}$ by

$$f(x) := \frac{d(x,B)}{d(x,B) + d(x,A)}.$$

Show that f is continuous on $\mathbf{M}$ and that $f(A) = \{1\}$, $f(B) = \{0\}$; i.e., f is identically 1 on A and identically 0 on B. Deduce that there are open sets $U, V \subset \mathbf{M}$ such that $A \subset U$, $B \subset V$, and $U \cap V = \emptyset$.

Answer: First, we must make sure that the function $f(x)$ is well-defined, which amounts to showing that the denominator is never 0. If $d(x, S) = 0$ for any set S, then $x \in S^-$. So if $d(x, A) = d(x, B) = 0$, then $x \in A^- \cap B^-$. However, we are told that A and B are closed, so we would conclude that $x \in A \cap B$. Because $A \cap B = \emptyset$, this cannot happen.

That settled, we have a quotient of continuous functions, so $f(x)$ is continuous. The interval $(\frac{2}{3}, \frac{4}{3})$ is open in $\mathbf{R}$, so $U := f^{-1}(\frac{2}{3}, \frac{4}{3})$ is open in $\mathbf{M}$. If $x \in A$ it is clear that $f(x) = 1$, so $A \subset U$. The interval $(-\frac{1}{3}, \frac{1}{3})$ is also open in $\mathbf{R}$, so $V := f^{-1}(-\frac{1}{3}, \frac{1}{3})$ is open in $\mathbf{M}$. If $x \in B$, we see easily that $f(x) = 0$, so $B \subset V$. Clearly, $U \cap V = \emptyset$.

2. **5.8.25.** Show that the metric space $(\mathbf{R}, d)$ where

$$d(x,y) := \left| \frac{x}{1+|x|} - \frac{y}{1+|y|} \right| \qquad (\forall x, y \in \mathbf{R}),$$

is not complete.

Answer: We will show that the sequence $(1, 2, 3, \ldots)$ of natural numbers is Cauchy and does not converge to a limit in $\mathbf{R}$. Given $\epsilon > 0$, find N so that $\frac{1}{N} < \frac{\epsilon}{2}$. Suppose that $m, n > N$. Suppose for simplicity that $m > n$. Then

$$d(m,n) = \left| \frac{m}{1+m} - \frac{n}{1+n} \right| = \left| \left(1 - \frac{1}{1+m}\right) - \left(1 - \frac{1}{1+n}\right) \right| = \left| \frac{1}{n+1} - \frac{1}{m+1} \right| < \frac{1}{n} + \frac{1}{m} < \epsilon.$$

This shows that the sequence is Cauchy.

Suppose that $\lim\limits_{n\to\infty} n = L$. Then given any ϵ, we can find some N so that if $n > N$, then $d(n, L) < \epsilon$. That says that

$$\left| \frac{L}{1+|L|} - \frac{n}{1+n} \right| < \epsilon$$

If $L < 0$, the difference will clearly be greater than $\frac{1}{2}$, so we may assume that $L \geq 0$ and drop the absolute value sign. We are then forced to conclude that

$$\left| \frac{1}{1+L} - \frac{1}{1+n} \right| < \epsilon$$

for all $n > N$. If we pick $\epsilon = \left(\frac{1}{3}\right) \frac{1}{1+L}$, and pick n so that $\frac{1}{1+n} < \frac{\epsilon}{2}$, we have a contradiction.

3. **5.8.37.** Show that the map $f(x) := x/(1+x)$ is a homeomorphism of $[0, \infty)$ onto $[0, 1)$. Deduce that, given any metric space $(\mathbf{M}, d)$, the metric $d' := d/(1+d)$ is equivalent to d.

Answer: First, note that there is a simpler way to do this problem. Using notation from an older problem, let $d_1 = d/(1+d)$, and let $d_2 = \min(1, d)$. We showed in that earlier problem that d_1 and d_2 are equivalent metrics. It is very simple to show that d and d_2 are equivalent metrics.

We set

$$B_\epsilon(x) = \{y \in \mathbf{M} : d(x,y) < \epsilon\}$$
$$B_\epsilon^{(1)}(x) = \{y \in \mathbf{M} : d_1(x,y) < \epsilon\}$$
$$B_\epsilon^{(2)}(x) = \{y \in \mathbf{M} : d_2(x,y) < \epsilon\}$$

Given any set U which is open with respect to d and a point $x \in U$, we know that for some $\epsilon > 0$, there is some $B_\epsilon(x) \subset U$. Let $\delta = \min(\epsilon, \frac{1}{2})$, and then $B_\delta(x) \subset U$. But $B_\delta(x) = B_\delta^{(2)}(x)$, because $\delta < 1$, showing that U is open with respect to d_2. The same argument shows that if U is open with respect to d_2, then it is open with respect to d.

It is not all that difficult to show directly that d is equivalent to d_1. Let $f(t) = t/(1+t)$. We know that f is a continuous increasing function, with continuous increasing inverse $g(s) = s/(1-s)$. Given any $\epsilon > 0$, let $\delta = f(\epsilon)$. Now, $B_\epsilon(x) = \{y : d(x,y) < \epsilon\} = \{y : \frac{d(x,y)}{1+d(x,y)} < \delta\} = \{y : d_1(x,y) < \delta\} = B_\delta^{(1)}(x)$.

Conversely, given any $\delta \in (0,1)$, let $\epsilon = g(\delta)$. Then $B_\delta^{(1)}(x) = \{y : d_1(x,y) < \delta\} = \{y : \frac{d(x,y)}{1+d(x,y)} < \delta\} = \{y : d(x,y) < \epsilon\} = B_\epsilon(x)$. Because open balls with respect to d and d_1 are the same, we know that d and d_1 are equivalent metrics.

4. **5.8.38.** In a metric space $(\mathbf{M}, d)$, let $A, B \subset \mathbf{M}$ be nonempty sets such that $A \cap B^- = B \cap A^- = \emptyset$. Show that there are open sets $U, V \subset \mathbf{M}$ with $A \subset U$, $B \subset V$, and $U \cap V = \emptyset$.

Answer: Let $f(x) = d(x,A) - d(x,B)$. We know that the two functions $d(x,A)$ and $d(x,B)$ are continuous, and that the difference of continuous functions is continuous, so $f(x)$ is continuous.

If $x \in A$, we know that $d(x,A) = 0$. Suppose that $x \in A$ and $d(x,B) = 0$. That implies that $x \in B^-$, and we know that $A \cap B^- = 0$. Hence if $x \in A$, we can conclude that $f(x) < 0$.

Conversely, suppose that $x \in B$. We know that $d(x,B) = 0$. If we also have $d(x,A) = 0$, that would imply that $x \in A^-$, contradicting the given information that $A^- \cap B = \emptyset$. So if $x \in B$, we can conclude that $f(x) > 0$.

The set $(-\infty, 0)$ is open in $\mathbf{R}$. If $U := f^{-1}(-\infty, 0)$, we know that U is open and $A \subset U$. The set $(0, \infty)$ is open in $\mathbf{R}$. If $V := f^{-1}(0, \infty)$, we know that V is open and $B \subset V$. Clearly, $U \cap V = \emptyset$.

5. **5.8.39.** Show that, given any metric space $(\mathbf{M}, d)$, the distance function $d : \mathbf{M} \times \mathbf{M} \to \mathbf{R}$ (given by $(x,y) \mapsto d(x,y)$) is uniformly continuous. Deduce that if $(x_n), (y_n) \in \mathbf{M}^{\mathbf{N}}$ are Cauchy sequences, then $(d(x_n, y_n))$ is a Cauchy sequence in $\mathbf{R}$.

Answer: Given $\epsilon > 0$, we need to find $\delta > 0$ so that if $d((u,v),(x,y)) < \delta$, then $|d(u,v) - d(x,y)| < \epsilon$. This problem is simplest if we use the metric d_{sum} on $\mathbf{M} \times \mathbf{M}$. In that case, $d_{\text{sum}}((u,v),(x,y)) = d(u,x) + d(v,y)$.

Now, given ϵ, let $\delta = \epsilon$. Suppose that $d(u,x) + d(v,y) < \epsilon$. Then $|d(u,v) - d(x,y)| \leq d(u,x) + d(v,y)$, so we are done.

6. **5.8.40.** Let $\mathbf{M}$ and $\mathbf{M}'$ be metric spaces. A property is said to be topological if it is preserved under homeomorphisms $f : \mathbf{M} \to \mathbf{M}'$. It is called a metric property if is is preserved under all bijective isometries $f : \mathbf{M} \to \mathbf{M}'$. Finally we call it a uniform property if it is preserved under uniform homeomorphisms, i.e., bijective maps $f : \mathbf{M} \to \mathbf{M}'$ such that both f and f^{-1} are uniformly continuous.

(a) Let $\mathbf{M} = (0,1] = \mathbf{M}'$, $d(x,y) := |x-y|$, and $d'(x,y) := |1/x - 1/y|$. Show that d and d' are equivalent, so that $\mathbf{M}$ and $\mathbf{M}'$ are homeomorphic. Show, however, that $(\mathbf{M}, d)$ is not complete, while $(\mathbf{M}', d')$ is. It follows that completeness is not a topological property. It is, however, a uniform property.

(b) Let $\mathbf{M}' = \mathbf{M}$, and $d' = d/(1+d)$. Show that d and d' are uniformly equivalent in the sense that the identity map $x \mapsto x$ is a uniform homeomorphism. Since d' is bounded while d need not be, deduce that boundedness is not a uniform property. However, it is a metric one.

(c) Let $\mathbf{M} = \mathbf{R} = \mathbf{M}'$, $d(x,y) := |x-y|$, and $d'(x,y) := |x^3 - y^3|$. Show that d and d' are equivalent but not uniformly equivalent. Nevertheless, show that $\mathbf{M}$ and $\mathbf{M}'$ have the same Cauchy sequences. Thus, completeness may be preserved without uniform equivalence.

Answer: I prefer to use $(\mathbf{M}_1, d_1)$ and $(\mathbf{M}_2, d_2)$ to label the metric spaces. In all three problems, $\mathbf{M}_1 = \mathbf{M}_2$. To show that these metric spaces are equivalent, we define $\iota : \mathbf{M}_1 \to \mathbf{M}_2$, $\iota(x) = x$, and show that ι and ι^{-1} are continuous.

To show that ι is continuous, we need to show that given any $x_0 \in \mathbf{M}_2$ and any $\epsilon > 0$, there is a $\delta > 0$ so that if $d_1(x, x_0) < \delta$, then $d_2(\iota(x), \iota(x_0)) < \epsilon$. The second inequality in fact is just $d_2(x, x_0) < \epsilon$. Similarly, to show that ι^{-1} is continuous, we need to show that given any $\epsilon > 0$, there is $\delta > 0$ so that if $d_2(x, x_0) < \delta$, then $d_1(x, x_0) < \epsilon$.

(a) Suppose that $\mathbf{M}_1 = (0,1] = \mathbf{M}_2$, with $d_1(x,y) = |x-y|$ and $d_2(x,y) = |x^{-1} - y^{-1}|$. Given $x_0 \in \mathbf{M}$ and $\epsilon > 0$, we need to show that there is some $\delta > 0$ so that if $|x - x_0| < \delta$, then $|x^{-1} - x_0^{-1}| < \epsilon$. But this is the same as the claim that the function $f : (0,1] \to [1,\infty)$, defined by $f(x) = x^{-1}$, is continuous, which we know is true. Therefore, ι is continuous.

Suppose that $\epsilon > 0$. We need to find $\delta > 0$ so that if $|x^{-1} - x_0^{-1}| < \delta$, then $|x - x_0| < \epsilon$. Now define $g : [1,\infty) \to (0,1]$ by $g(x) = x^{-1}$. We know that $g(x)$ is continuous, which shows that δ exists.

In $\mathbf{M}_1$, the sequence (n^{-1}) is Cauchy but converges to 0, which is not in $\mathbf{M}_1$. Therefore, $\mathbf{M}_1$ is not complete.

Consider the function $g(x)$ defined above. It is an isometry from the metric space $[1,\infty)$ with the usual metric $d(x,y) = |x-y|$ to $\mathbf{M}_2$. Isometries preserve completeness (because they map Cauchy sequences to Cauchy sequences), and we know that $[1,\infty)$ is complete. This shows that $\mathbf{M}_2$ is complete.

(b) Let $(\mathbf{M}_1, d_1) = (\mathbf{M}, d)$, and $(\mathbf{M}_2, d_2) = (\mathbf{M}, d/(1+d))$. We need to show that ι and ι^{-1} are uniformly continuous.

Given $\epsilon' > 0$, let $\epsilon = \min(\epsilon', \frac{1}{2})$. Find δ so that $\delta/(1+\delta) = \epsilon$. (This is why we had to replace ϵ' with ϵ. If $\epsilon' > 1$, it is not possible to find δ so that $\delta/(1+\delta) = \epsilon'$.) If $d_1(x,y) < \delta$, then $d_2(x,y) = d_1(x,y)/(1 + d_1(x,y)) < \epsilon$. Therefore, ι is uniformly continuous. Given $\epsilon > 0$, let $\delta = \epsilon/(1+\epsilon)$. If $d_2(x,y) < \delta$, then $d_1(x,y) < \epsilon$, showing that ι^{-1} is uniformly continuous.

(c) Let $\mathbf{M}_1 = \mathbf{R} = \mathbf{M}_2$ with $d_1(x,y) = |x-y|$, and $d_2(x,y) = |x^3 - y^3|$. We must show that ι and ι^{-1} are continuous. To show that ι is continuous, we must show that given any $\epsilon > 0$ and $x_0 \in \mathbf{R}$, there is $\delta > 0$ so that if $d_1(x, x_0) < \delta$, then $d_2(x, x_0) < \epsilon$. In other words, we must find δ so that if $|x - x_0| < \delta$, then $|x^3 - x_0^3| < \epsilon$. This is the same as showing that the function $f(x) = x^3$ is continuous, which we know.

To show that ι^{-1} is continuous, we must show that given any $\epsilon > 0$ and $x_0 \in \mathbf{R}$, we can find $\delta > 0$ so that if $|x^3 - x_0^3| < \delta$, then $|x - x_0| < \epsilon$. Let $y = x^{1/3}$ and $y_0 = x_0^{1/3}$. We need to show that if $|y - y_0| < \delta$, then $|y^{1/3} - y_0^{1/3}| < \epsilon$. This is the same as showing that the function $g(y) = y^{1/3}$ is continuous, which we know.

Next, we show that ι is not uniformly continuous. The only way to show such a thing is by contradiction. Suppose that for every $\epsilon > 0$, there is $\delta > 0$ so that if $|x - y| < \delta$, then $|x^3 - y^3| < \epsilon$. In particular, take $\epsilon = 1$. Suppose that such a δ exists. Let y be a large positive number, and $x = y + \frac{\delta}{3}$. Then $|x - y| < \delta$. However, $|x^3 - y^3| = (y + \frac{\delta}{3})^3 - y^3 > y^2\delta$. Given any δ, we can always find a large positive number y so that $y^2\delta > 1$, showing that ι is not uniformly continuous.

Finally, we must show that ι and ι^{-1} preserve Cauchy sequences. Suppose first that (x_n) is Cauchy in $\mathbf{M}_1$. Because $\mathbf{M}_1$ is complete, we can find x so that $x_n \to x$. Then we know that $x_n^3 \to x^3$, showing that (x_n) also converges to x in $\mathbf{M}_2$.

Suppose instead that (x_n) is Cauchy in MM_2. That says that the sequence (x_n^3) is Cauchy in the usual metric d_1 on $\mathbf{R}$, so we can find some y with $x_n^3 \to y$. Then, again with respect to the usual metric d_1, $x_n \to y^{1/3}$, showing that (x_n) is Cauchy with respect to d_1.