Mathematics 805
Homework 5
Due Friday, February 20, 1 PM

1. **5.6.3.** Let $(K, d)$ be a compact metric space. Then $(C(K), d_\infty)$, where $d_\infty$ is the uniform metric, is complete.

*Answer:* Let $(f_k)$ be a Cauchy sequence in $C(K)$. Given any $\epsilon > 0$, we can find $N$ so that if $m, n > N$, then $d_\infty(f_m, f_n) < \epsilon$. If $k \in K$ is some fixed element, then $d(f_m(k), f_n(k)) \leq d_\infty(f_m, f_n)$, and therefore the sequence $(f_m(k))$ is a Cauchy sequence of real numbers, which therefore converges to some limit. Define the function $f(k)$ by $f(k) = \lim_{n\to\infty} f_n(k)$. We need to show both that $f$ is continuous and that $f = \lim_{n\to\infty} f_n$. It is actually easier to show the second result.

Given any $\epsilon > 0$, find some $N$ so that if $m, n > N$, then $d_\infty(f_m, f_n) < \frac{\epsilon}{2}$. In particular, this says that $|f_m(k) - f_n(k)| < \frac{\epsilon}{2}$ for all $k \in K$ and all $m, n > N$. Taking a limit as $n \to \infty$, we see that $|f_m(k) - f(k)| \leq \frac{\epsilon}{2}$. Because this is true for all $k \in K$, we have $d_\infty(f_m, f) \leq \frac{\epsilon}{2}$ for all $m > N$, and so $d_\infty(f_m, f) < \epsilon$. This shows that $f = \lim f_n$.

Now, we can use a standard $\frac{\epsilon}{3}$ argument to show that $f$ is continuous. Given $x \in K$ and $\epsilon > 0$, find $n$ so that $d(f, f_n) < \frac{\epsilon}{3}$. Because $f_n \in C(K)$, there is $\delta > 0$ so that if $d(k', k) < \delta$, then $|f_n(k') - f_n(k)| < \frac{\epsilon}{3}$. Then $|f(k') - f(k)| \leq |f(k') - f_n(k')| + |f_n(k') - f_n(k)| + |f_n(k) - f(k)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$.

2. **5.6.5.** Let $A$ and $B$ be nonempty subsets of a metric space $(\mathbf{M}, d)$.

($a$) Show that if $A$ is compact, there there is a point $a \in A$ so that $d(a, B) = d(A, B)$.

($b$) Show that if $A$ and $B$ are both compact, then there exist $a \in A$ and $b \in B$ so that $d(a, b) = d(A, B)$.

($c$) Show that if $A$ is compact and $B$ is closed, then $d(A, B) = 0$ if and only if $A \cap B \neq \emptyset$.

*Answer:* ($a$) We need to show that the function $f(x) = d(x, B)$ is continuous. Given any $\epsilon > 0$, suppose that $d(x, y) < \epsilon$. Then for any point $b \in B$, $d(x, b) \leq d(x, y) + d(y, b)$, and so $d(x, B) = \inf d(x, b) \leq d(x, y) + d(y, b) < \epsilon + d(y, b)$. Summarizing, $f(x) < \epsilon + d(y, b)$ for all $b \in B$, and therefore $f(x) \leq \epsilon + f(y)$, or $f(x) - f(y) \leq \epsilon$. We can reverse the roles of $x$ and $y$ and get the same inequality, showing that $|f(x) - f(y)| \leq \epsilon$.

Now that we see that $f(x)$ is continuous, the rest is easy. The function $f(x)$ is a continuous function on a compact domain, so it attains its minimum, which is $f(A, B)$.

($b$) Find $a$ so that $d(a, B) = d(A, B)$, and then the function $g(b) = d(a, b)$ is continuous and also attains its minimum on the compact set $B$.

($c$) If $A \cap B \neq \infty$, then obviously $d(A, B) = 0$. So suppose that $d(A, B) = 0$, with $A$ compact and $B$ closed. By ($a$), we can find $a \in A$ with $d(a, B) = d(A, B)$, so we have $a \in A$ with $d(a, B) = 0$. That means that for any $\epsilon > 0$, we can find $b \in B$ with $d(a, b) < \epsilon$. That says that $a$ is a limit point of $B$. Because $B$ is closed, it contains its limit points, and therefore $a \in B$, showing that $A \cap B \neq \emptyset$.

3. **5.8.43.** Let $K$ be a compact subset of a metric space $(\mathbf{M}, d)$. Show that there are two points $x, y \in K$ so that $\delta(K) = d(x, y)$.

*Answer:* By definition $\delta(K) = \sup_{x,y\in K} d(x, y)$. The function $d(x, y)$ is continuous on $K \times K$, and $K \times K$ is a compact set; therefore, it attains its maximum value.

4. **5.8.54.** Let $(K, d)$ be a compact metric space, and let $f : K \to K$ be an isometry, i.e., $d(f(x), f(y)) = d(x, y)$ for all $x, y \in K$. Show that $f$ is onto. Give an example to show that if $K$ is not compact, then $f$ need not be onto.

*Answer:* Suppose that $f$ is not onto. We know that $f$ is closed, so $f(K)$ is a closed subset of $K$. Therefore, $f(K)^c$ is open. Suppose that $x_0 \in f(K)^c$. Then for some $\epsilon > 0$, $B_\epsilon(x_0) \subset f(K)^c$, meaning that if $y \in f(K)$, then $d(x_0, y) > \epsilon$.

Let $x_1 = f(x_0)$. Because $x_1 \in f(K)$, we know that $d(x_0, x_1) > \epsilon$.

Let $x_2 = f(x_1)$. Then we know that $d(x_0, x_2) > \epsilon$. Also, $d(x_1, x_2) = d(f(x_0), f(x_1)) = d(x_0, x_1) > \epsilon$. So in the set $A_2 = \{x_0, x_1, x_2\}$, $d(x_i, x_j) > \epsilon$.

We now inductively define $A_n$ by $x_n = f(x_{n-1})$ and $A_n = A_{n-1} \cup \{x_n\}$. We can see that if $x_i, x_j \in A_n$, then $d(x_i, x_j) > \epsilon$. Because this assertion is true for elements of $A_{n-1}$, we need only check that $d(x_i, x_n) > \epsilon$.

If $i = 0$, this is true because $x_n \in f(K)$. If $i > 0$, we have $d(x_i, x_n) = d(f(x_{i-1}, f(x_{n-1})) = d(x_{i-1}, x_{n-1}) > \epsilon$ because $x_{i-1}, x_{n-1} \in A_{n-1}$.

Now, the set $A = \bigcup A_n$ is a discrete set, and hence has no limit points. This shows that $K$ is not Fréchet compact, which is a contradiction.

If $A$ is not compact, the result need not be true. Consider the set $A = (0, \infty) \subset \mathbf{R}$, and the function $f : A \to A$ defined by $f(x) = x + 1$. This is an isometry which is not surjective.

5. **5.8.55.** Let $(K, d)$ and $(K', d')$ be two compact metric spaces and let $f : K \to K'$ and $g : K' \to K$ be isometries. Show that $f(K) = K'$ and $g(K') = K$.

*Answer:* We see that $g \circ f : K \to K$ is an isometry, so we know from the previous problem that $g \circ f(K) = K$. If $g \circ f$ is surjective, then $g$ must be surjective. Apply the same argument to $f \circ g$ to see that $f$ is surjective.