Mathematics 805
Homework 6
Due Friday, March 13, 1 PM

1. **8.7.3.** Show that $(\sin nx/(1+nx))$ converges uniformly on $[a,\infty)$ for any $a>0$ but not for $a=0$.

Answer: Let $f_n(x)=\sin nx/(1+nx)$. Because $|f_n(x)|<1/(1+nx)$, we see that if $x\neq 0$, then $\lim f_n(x)=0$. Of course, $f_n(0)=0$, so we have $\lim f_n(x)=0$ for all values of x.

Now, if $x\in[a,\infty)$, we have $|f_n(x)|<1/(1+na)$. Given $\epsilon>0$, we can take N large enough so that $1/(1+Na)<\epsilon$, and then we have $|f_n(x)|<\epsilon$ for all $n>N$ for all $x\in[a,\infty)$.

Suppose instead we consider $x\in[0,\infty)$. For any positive integer n, we can let $x=\frac{1}{n}$, and we have $|f_n(\frac{1}{n})|=(\sin 1)/2>0.4$. This shows that $f_n(x)$ does not tend to 0 uniformly on $[0,\infty)$.

2. **8.7.4.** Let $f_n(x)=(1+x/n)^n$ for all $x\in\mathbf{R}$ and $n\in\mathbf{N}$. Show that $f_n(x)$ converges uniformly to e^x on any compact interval $[a,b]\subset\mathbf{R}$.

Answer: First, we need to show that $f_n(x)\to e^x$. Write $f_n(x)=\exp(n\log(1+x/n))=\exp(n\log(\frac{n+x}{n}))=\exp(n(\log(n+x)-\log n))$. Now, applying l'Hôpital's Rule, we have

$$\lim_{n\to\infty} n(\log(n+x)-\log n)=\lim_{n\to\infty}\frac{\log(n+x)-\log n}{n^{-1}}=\lim_{n\to\infty}\frac{\frac{1}{n+x}-\frac{1}{n}}{-n^{-2}}=\lim_{n\to\infty}\frac{\frac{-x}{n(n+x)}}{-n^{-2}}=\lim_{n\to\infty}\frac{n^2x}{n^2+nx}=x,$$

and so $\lim\limits_{n\to\infty}\exp(n(\log(n+x)-\log n))=e^x$.

It remains to show that the convergence is uniform on any compact interval $[a,b]$. To do that, we can use Dini's Theorem (Theorem 8.2.2), and show that $f_n(x)\le f_{n+1}(x)$. We apply the binomial theorem to each function:

$$\begin{aligned}
f_n(x)&=\left(1+\frac{x}{n}\right)^n=1+n\left(\frac{x}{n}\right)+\frac{n(n-1)}{2}\left(\frac{x}{n}\right)^2+\frac{n(n-1)(n-2)}{6}\left(\frac{x}{n}\right)^3+\cdots\\
&=1+x+\frac{1}{2}\left(1\cdot\left(1-\frac{1}{n}\right)\right)x^2+\frac{1}{6}\left(1\cdot\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\right)x^3+\cdots\\
f_{n+1}(x)&=\left(1+\frac{x}{n+1}\right)^{n+1}=1+(n+1)\left(\frac{x}{n+1}\right)+\frac{(n+1)n}{2}\left(\frac{x}{n+1}\right)^2+\frac{(n+1)n(n-1)}{6}\left(\frac{x}{n+1}\right)^3+\cdots\\
&=1+x+\frac{1}{2}\left(1\cdot\left(1-\frac{1}{n+1}\right)\right)x^2+\frac{1}{6}\left(1\cdot\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\right)x^3+\cdots
\end{aligned}$$

We see that term by term, the monomials in the expansion of $f_{n+1}(x)$ are at least as large as those in the expansion of $f_n(x)$, showing that $f_n(x)\le f_{n+1}(x)$. Note that this argument only works when $x>0$; a more subtle argument is needed for $x<0$.

3. **8.7.12.** For each $n\in\mathbf{N}$, consider the function

$$f_n(x):=\frac{x^n}{1+x^{2n}}\quad(\forall x\in\mathbf{R})$$

(a) Show that the sequence (f_n) converges uniformly on $[a,b]$ if and only if $|x|\neq 1$ for all $x\in[a,b]$ (i.e., $[a,b]$ does not contain the points 1 and -1).

(b) Find all $x\in\mathbf{R}$ for which the series $\sum f_n(x)$ is convergent. Also, find the intervals on which the convergence is uniform.

Answer: Note that $f_n(x)=f_n(1/x)$, a fact which we will exploit momentarily.

If $x=1$, then $f_n(x)=\frac{1}{2}$, so $\lim\limits_{n\to\infty}f_n(1)=\frac{1}{2}$. If $x=-1$, then $f_{2n}(-1)=\frac{1}{2}$, while $f_{2n+1}(-1)=-\frac{1}{2}$, so $\lim\limits_{n\to\infty}f_n(-1)$ does not exist. If $|x|<1$, then $\left|\frac{x^n}{1+x^{2n}}\right|<|x^n|$, so $\lim\limits_{n\to\infty}f_n(x)=0$. If $|x|>1$, then $|1/x|<1$, so $\lim\limits_{n\to\infty}f_n(x)=\lim\limits_{n\to\infty}f_n(1/x)=0$.

Suppose that $[a,b]$ does not contain ± 1. If $|a| < 1$ and $|b| < 1$, and $x \in [a,b]$, then $|f_n(x)| < |x^n| < \max(|a|^n, |b|^n)$. Given $\epsilon > 0$, choose N so that $|a|^N < \epsilon$ and $|b|^N < \epsilon$. If $n > N$, then $|f_n(x)| < \epsilon$.

If $|a| > 1$ and $|b| > 1$, $|f_n(x)| < \max(|a|^{-n}, |b|^{-n})$. Given $\epsilon > 0$, choose N so that $|a|^{-N} < \epsilon$ and $|b|^{-N} < \epsilon$. If $n > N$, then $|f_n(x)| < \epsilon$.

4. **8.7.22.** Prove each equation.

(a) $\displaystyle\int_1^2 \left(\sum_{n=1}^{\infty} ne^{-nx}\right) dx = \frac{e}{e^2-1}$ $\qquad$ (b) $\displaystyle\int_0^{\pi} \left(\sum_{n=1}^{\infty} \frac{n\sin nx}{e^n}\right) dx = \frac{2e}{e^2-1}.$

Answer: (a) First, note that if $x \in [1,2]$, then $ne^{-nx} < 2e^{-n}$. Because $\sum_{n=1}^{\infty} 2e^{-n}$ converges, the Weierstrass M-test shows that $\sum_{n=1}^{\infty} ne^{-nx}$ converges uniformly on $[1,2]$, and therefore we can interchange the order of integration and summation. We have

$$\int_1^2 \left(\sum_{n=1}^{\infty} ne^{-nx}\right) dx = \sum_{n=1}^{\infty} \left(\int_1^2 ne^{-nx}\, dx\right) = \sum_{n=1}^{\infty} -e^{-nx}\Big|_{x=1}^{2}$$
$$= \sum_{n=1}^{\infty} \left(e^{-n} - e^{-2n}\right) = \frac{e^{-1}}{1-e^{-1}} - \frac{e^{-2}}{1-e^{-2}} = \frac{1}{e-1} - \frac{1}{e^2-1} = \frac{e}{e^2-1}.$$

(b) Because $|\frac{n\sin nx}{e^n}| < ne^{-n}$, and $\sum ne^{-n}$ converges, we can apply the Weierstrass M-test to conclude that $\sum_{n=1}^{\infty} \frac{n\sin nx}{e^n}$ converges uniformly, ande hence we can interchange the orders of integration and summation. We have

$$\int_0^{\pi} \left(\sum_{n=1}^{\infty} \frac{n\sin nx}{e^n}\right) dx = \sum_{n=1}^{\infty} \left(\int_0^{\pi} \frac{n\sin nx}{e^n}\, dx\right) = \sum_{n=1}^{\infty} \left(\frac{-\cos nx}{e^n}\Big|_{x=0}^{\pi}\right) = \sum_{n=1}^{\infty} \left(\frac{1-\cos(n\pi)}{e^n}\right)$$
$$= \sum_{n=1}^{\infty} \frac{2}{e^{2n-1}} = \frac{2e^{-1}}{1-e^{-2}} = \frac{2e}{e^2-1}$$

5. **8.7.24.** Let $f_n(x) := x/(1+n^2x^2)$ for all $x \in [-1,1]$. Show that (f_n) converges uniformly to a differentiable function f, but that $f_n'(x)$ does not converge to $f'(x)$ for all $x \in [-1,1]$.

Answer: Note that $\lim_{n\to\infty} f_n(0) = 0$. If $x \neq 0$, then $|f_n(x)| < \frac{|x|}{n^2x^2} = \frac{1}{n^2|x|}$, so we still have $\lim_{n\to\infty} f_n(x) = 0$. We need to show that the convergence is uniform.

Given $\epsilon > 0$, note that if $|x| < \epsilon$, then $|f_n(x)| < |x| < \epsilon$ for all n. For $|x| > \epsilon$, pick N so that $\frac{1}{N^2\epsilon} < \epsilon$, i.e., $N^2 > \frac{1}{\epsilon^2}$ or $N > \frac{1}{\epsilon}$.

We have $|f_n(x)| < \frac{|x|}{n^2x^2} = \frac{1}{n^2|x|} \leq \frac{1}{n^2\epsilon} \leq \frac{1}{N^2\epsilon} < \epsilon$. This shows that $f_n(x)$ converges to 0 uniformly on $[-1,1]$.

We have $f_n'(x) = \frac{1+n^2x^2-2n^2x^2}{(1+n^2x^2)^2} = \frac{1-n^2x^2}{(1+n^2x^2)^2}$. We see that $f_n'(0) = 1$ for all n, while if $x \neq 0$, $f_n'(x) \to 0$.