Mathematics 805
Homework 7
Due Friday, March 20, 1 PM

1. **8.7.25.** For each nonnegative integer n, define the n-th order Bessel function by

$$J_n(x) := \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!} \left(\frac{x}{2}\right)^{2k+n}.$$

(a) Show that $J_n(x)$ converges pointwise on $\mathbf{R}$ and uniformly on any $[a,b] \subset \mathbf{R}$.
(b) Show that $(x^n J_n(x))' = x^n J_{n-1}(x)$ for all $x \in \mathbf{R}$ and every positive integer n.
(c) Show that $y := J_n(x)$ satisfies Bessel's differential equation

$$x^2 y'' + xy' + (x^2 - n^2)y = 0.$$

Answer: (a) We can apply the ratio test to see that the series converges for all x. The ratio of the absolute value of consecutive terms is:

$$\frac{1}{(k+1)!(k+n+1)!}\left(\frac{x}{2}\right)^{2k+n+2} \Big/ \frac{1}{k!(k+n)!}\left(\frac{x}{2}\right)^{2k+n} = \frac{x^2}{(k+1)(k+n+1)2^2}$$

which tends to 0 as k tends to infinity for all values of x. This shows that the series converges for all x, and therefore it converges uniformly on any $[a,b] \subset \mathbf{R}$.

It is possible that the exercise is asking instead about the pointwise convergence of $\lim_{n\to\infty} J_n(x)$ for fixed x. This is easily done:

$$|J_n(x)| = \left|\sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!}\left(\frac{x}{2}\right)^{2k+n}\right| \le \frac{|x|^n}{n!}\sum_{k=0}^{\infty}\frac{(x^2/4)^k}{k!} = \frac{|x|^n}{n!}e^{x^2/4}$$

For fixed x, we know that $\frac{|x|^n}{n!} \to 0$, and therefore $\lim J_n(x) = 0$. Moreover, if we restrict to $x \in [a,b]$, and define $c = \max(|a|,|b|)$, we have $|x|^n e^{x^2/4} \le c^n e^{c^2/4}$, showing that the convergence is uniform on $[a,b]$.

(b) We have

$$x^n J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+2n}}{k!(k+n)!2^{2k+n}}$$

and so

$$\frac{d}{dx}(x^n J_n(x)) = \sum_{k=0}^{\infty}\frac{(2k+2n)(-1)^k x^{2k+2n-1}}{k!(k+n)!2^{2k+n}} = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+2n-1}}{k!(k+n-1)!2^{2k+n-1}}$$
$$= x^n \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+n-1}}{k!(k+n-1)!2^{2k+n-1}} = x^n J_{n-1}(x)$$

(c) We have

$$y = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!(k+n)!}\left(\frac{x}{2}\right)^{2k+n}$$
$$y' = \sum_{k=0}^{\infty}\frac{(-1)^k(2k+n)}{k!(k+n)!}\left(\frac{x^{2k+n-1}}{2^{2k+n}}\right)$$
$$xy' = \sum_{k=0}^{\infty}\frac{(-1)^k(2k+n)}{k!(k+n)!}\left(\frac{x^{2k+n}}{2^{2k+n}}\right)$$
$$y'' = \sum_{k=0}^{\infty}\frac{(-1)^k(2k+n)(2k+n-1)}{k!(k+n)!}\left(\frac{x^{2k+n-2}}{2^{2k+n}}\right)$$
$$x^2y'' = \sum_{k=0}^{\infty}\frac{(-1)^k(2k+n)(2k+n-1)}{k!(k+n)!}\left(\frac{x^{2k+n}}{2^{2k+n}}\right)$$

and so

$$
\begin{aligned}
x^2y'' + xy' &= \sum_{k=0}^{\infty} \left(\frac{(-1)^k (2k+n)(2k+n-1)}{k!(k+n)!} + \frac{(-1)^k(2k+n)}{k!(k+n)!} \right) \left(\frac{x^{2k+n}}{2^{2k+n}} \right) \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k(2k+n)^2}{k!(k+n)!} \left(\frac{x^{2k+n}}{2^{2k+n}} \right) \\
&= n^2 \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!} \left(\frac{x^{2k+n}}{2^{2k+n}} \right) + \sum_{k=0}^{\infty} \frac{(-1)^k(4kn+4k^2)}{k!(k+n)!} \left(\frac{x^{2k+n}}{2^{2k+n}} \right) \\
&= n^2 y + \sum_{k=0}^{\infty} \frac{(-1)^k(4kn+4k^2)}{k!(k+n)!} \left(\frac{x^{2k+n}}{2^{2k+n}} \right) \\
&= n^2 y + \sum_{k=1}^{\infty} \frac{(-1)^k(4n+4k)}{(k-1)!(k+n)!} \left(\frac{x^{2k+n}}{2^{2k+n}} \right) \\
&= n^2 y + \sum_{k=0}^{\infty} \frac{(-1)^{k+1}(4n+4k+4)}{k!(k+n+1)!} \left(\frac{x^{2k+n+2}}{2^{2k+n+2}} \right) \\
&= n^2 y - x^2 \sum_{k=0}^{\infty} \frac{(-1)^k(n+k+1)}{k!(k+n+1)!} \left(\frac{x^{2k+n}}{2^{2k+n}} \right) \\
&= n^2 y - x^2 \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!} \left(\frac{x^{2k+n}}{2^{2k+n}} \right) = n^2 y - x^2 y
\end{aligned}
$$

We have $x^2y'' + xy' = (n^2 - x^2)y$, which is the desired relationship.

2. **8.7.26.** For each power series, find the radius of convergence and determine whether the series converges at the endpoints of the interval of convergence.

(*a*) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ (*b*) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{\log(1+n)}$ (*c*) $\displaystyle\sum_{n=1}^{\infty} 2^n n^2 x^n$

(*d*) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n} x^n$ (*e*) $\displaystyle\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)^n x^n$ (*f*) $\displaystyle\sum_{n=1}^{\infty} \left(-\frac{1}{3}\right)^n x^n$

Answer: (*a*) Because $\lim (n+1)^2/n^2 = 1$, we know immediately that $R = 1$. When $x = 1$, we get the series $\sum n^{-2}$, which converges via the integral test. When $x = -1$, we get the series $\sum (-1)^n n^{-2}$, which converges via the alternating series test.

(*b*) This time, we need to compute $\lim \log(2+n)/\log(1+n)$, which by l'Hôpital's Rule is 1. Therefore, $R = 1$. When $x = 1$, we have $\sum (-1)^n/\log(n+1)$, which converges via the alternating series test. When $x = -1$, we have $\sum 1/\log(n+1)$. Because $1/\log(n+1) > 1/(n+1)$, we know that this series diverges.

(*c*) We compute $\lim (2^n n^2)/(2^{n+1}(n+1)^2) = \frac{1}{2}$, so $R = \frac{1}{2}$. When $x = \frac{1}{2}$, we have $\sum n^2$, which diverges because the terms do not tend to 0. When $x = -\frac{1}{2}$, we have $\sum (-1)^n n^2$, which also diverges because the terms do not tend to 0.

(*d*) We compute

$$\lim \frac{n!/n^n}{(n+1)!/(n+1)^{n+1}} = \lim \frac{(n+1)^{n+1}}{(n+1)n^n} = \lim \frac{(n+1)^n}{n^n} = \lim \left(1 + \frac{1}{n}\right)^n = e.$$

Therefore, $R = e$. To decide what happens when $x = \pm e$, we need to use a rather sharp form of Stirling's Formula, which states that $n! > \sqrt{2\pi n}\frac{n^n}{e^n}$. Therefore, when we substitute $x = e$ into the series, we have

$$\sum \frac{n! e^n}{n^n} > \sum \sqrt{2\pi n} \left(\frac{n^n}{e^n} \right) \left(\frac{e^n}{n^n} \right) = \sum \sqrt{2\pi n}$$

This sum diverges, because the terms do not go to 0, and therefore the original series diverges when $x = e$. Similarly, the series diverges when $x = -e$ because we get an alternating series whose terms do not tend to 0.

(e) We use the root test, and compute

$$\limsup |(1-1/n)^n|^{1/n} = \limsup(1-1/n) = 1.$$

Therefore, $R = 1$. Because $\lim(1-1/n)^n = e^{-1} \neq 0$, we see that the series does not converge when $x = \pm 1$.

(f) We see using the root test that $R = 3$, and that the series diverges when $x = \pm 3$, because the terms do not tend to 0.

3. **8.7.31.** Using the fact that $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n$ on $(-1,1)$, find the sums of the following series:

(a) $\displaystyle\sum_{n=1}^{\infty} nx^n$ (b) $\displaystyle\sum_{n=1}^{\infty} n^2x^n$ (c) $\displaystyle\sum_{n=0}^{\infty} n^3x^n$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{nx^n}{n+1}$ (e) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n(n+1)}$ (f) $\displaystyle\sum_{n=0}^{\infty} \frac{nx^n}{(n+1)(n+2)}$

Answer: The given power series converges uniformly on any interval $[a,b]$ with $-1 < a < b < 1$. This justifies term-by-term differentiation and integration in this interval.

(a) We have

$$\begin{aligned}
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n \\
\frac{d}{dx}\left(\frac{1}{1-x}\right) &= \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = \sum_{n=0}^{\infty} \frac{d}{dx}(x^n) \\
\left(\frac{1}{1-x}\right)^2 &= \sum_{n=1}^{\infty} nx^{n-1} \\
\frac{x}{(1-x)^2} &= \sum_{n=1}^{\infty} nx^n
\end{aligned}$$

(b) Continuing, we have

$$\begin{aligned}
\frac{x}{(1-x)^2} &= \sum_{n=1}^{\infty} nx^n \\
\frac{d}{dx}\left(\frac{x}{(1-x)^2}\right) &= \frac{d}{dx}\left(\sum_{n=1}^{\infty} nx^n\right) = \sum_{n=1}^{\infty} \frac{d}{dx}(nx^n) \\
\frac{x+1}{(1-x)^3} &= \sum_{n=1}^{\infty} n^2x^{n-1} \\
\frac{x^2+x}{(1-x)^3} &= \sum_{n=1}^{\infty} n^2x^n
\end{aligned}$$

(c) Continuing, we have

$$\begin{aligned}
\frac{x^2+x}{(1-x)^3} &= \sum_{n=1}^{\infty} n^2x^n \\
\frac{d}{dx}\left(\frac{x^2+x}{(1-x)^3}\right) &= \frac{d}{dx}\left(\sum_{n=1}^{\infty} n^2x^n\right) = \sum_{n=1}^{\infty} \frac{d}{dx}(n^2x^n) \\
\frac{x^2+4x+1}{(x-1)^4} &= \sum_{n=1}^{\infty} n^3x^{n-1} \\
\frac{x^3+4x^2+x}{(x-1)^4} &= \sum_{n=1}^{\infty} n^3x^n
\end{aligned}$$

(d) We have

$$\begin{aligned}
\frac{x}{(1-x)^2} &= \sum_{n=1}^{\infty} nx^n \\
\int_0^t \frac{x\,dx}{(1-x)^2} &= \int_0^t \left(\sum_{n=1}^{\infty} nx^n\right) dt = \sum_{n=1}^{\infty} \int_0^t nx^n\,dt \\
\log(1-x) + \frac{1}{1-x}\Big]_0^t &= \sum_{n=1}^{\infty} \frac{nx^{n+1}}{n+1}\Big]_0^t \\
\log(1-t) + \frac{1}{1-t} - 1 &= \sum_{n=1}^{\infty} \frac{nt^{n+1}}{n+1} \\
\log(1-t) + \frac{t}{1-t} &= \sum_{n=1}^{\infty} \frac{nt^{n+1}}{n+1} \\
\frac{\log(1-t)}{t} + \frac{1}{1-t} &= \sum_{n=1}^{\infty} \frac{nt^n}{n+1}
\end{aligned}$$

(e) We have

$$\begin{aligned}
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n \\
\int_0^t \frac{dx}{1-x} &= \int_0^t \left(\sum_{n=0}^{\infty} x^n\right) dx = \sum_{n=0}^{\infty} \int_0^t x^n\,dx \\
-\log(1-x)\Big]_0^t &= \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}\Big]_0^t \\
-\log(1-t) &= \sum_{n=0}^{\infty} \frac{t^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{t^n}{n} \\
-\int_0^y \log(1-t)\,dt &= \int_0^y \left(\sum_{n=1}^{\infty} \frac{t^n}{n}\right) dt = \sum_{n=1}^{\infty} \int_0^y \frac{t^n}{n}\,dt \\
(1-t)\log(1-t) - (1-t)\Big]_0^y &= \sum_{n=1}^{\infty} \frac{t^{n+1}}{n(n+1)}\Big]_0^y \\
(1-y)\log(1-y) + y &= \sum_{n=1}^{\infty} \frac{y^{n+1}}{n(n+1)} \\
\frac{(1-y)\log(1-y) + y}{y} &= \sum_{n=1}^{\infty} \frac{y^n}{n(n+1)}
\end{aligned}$$

(f) We have

$$\begin{aligned}
\log(1-t) + \frac{t}{1-t} &= \sum_{n=1}^{\infty} \frac{nt^{n+1}}{n+1} \\
\int_0^y \left(\log(1-t) + \frac{t}{1-t}\right) dt &= \int_0^y \left(\sum_{n=1}^{\infty} \frac{nt^{n+1}}{n+1}\right) dt = \sum_{n=1}^{\infty} \int_0^y \frac{nt^{n+1}}{n+1}\,dt \\
(t-2)\log(1-t) - 2t\Big]_0^y &= \sum_{n=1}^{\infty} \frac{nt^{n+2}}{(n+1)(n+2)}\Big]_0^t \\
(y-2)\log(1-y) - 2y &= \sum_{n=1}^{\infty} \frac{ny^{n+2}}{(n+1)(n+2)}
\end{aligned}$$

$$\frac{(y-2)\log(1-y)-2y}{y^2} = \sum_{n=1}^{\infty} \frac{ny^n}{(n+1)(n+2)}$$

4. **8.7.33.** Show that

$$\log\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{x^{2n+1}}{2n+1} + \cdots\right) \qquad (|x| < 1).$$

Answer: Integrating $\frac{1}{1+x} = \sum(-1)^n x^n$ and $\frac{1}{1-x} = \sum x^n$ from 0 to t, we get

$$\log(1+t) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}t^n}{n}$$
$$-\log(1-t) = \sum_{n=1}^{\infty} \frac{t^n}{n}$$

Because both series converge absolutely inside $(-1,1)$, we can add and rearrange to get

$$\log\left(\frac{1+t}{1-t}\right) = \sum_{n=1}^{\infty}(1+(-1)^{n-1})\frac{t^n}{n} = \sum_{n=0}^{\infty} \frac{2t^{2n+1}}{2n+1}.$$