Mathematics 805
Homework 8
Due Friday, March 27, 1 PM

1. A good reference for the history of factorials and $\Gamma(x)$ is an article by P.J. Davis in *The American Mathematical Monthly*, vol. 66, 1959, pp. 849–869, available through `www.jstor.org`.

This proof of a less precise form of Stirling's formula is taken from Rudin's *Principles of Mathematical Analysis*.

For m a positive integer, define

$$\begin{aligned} f(x) &= (m+1-x)\log m + (x-m)\log(m+1) && m \le x \le m+1 \\ g(x) &= \frac{x}{m} - 1 + \log m && m - \tfrac{1}{2} \le x < m + \tfrac{1}{2} \end{aligned}$$

You might find it helpful to graph $f(x)$ and $g(x)$ to see why one might define such functions.

Prove that

$$f(x) \le \log x \le g(x).$$

This shows that if n is a positive integer,

$$\int_1^n f(x)\,dx \le \int_1^n \log x\,dx \le \int_1^n g(x)\,dx.$$

Use this inequality to show that

$$\tfrac{7}{8} < \log(n!) - (n + \tfrac{1}{2})\log n + n < 1,$$

thereby proving that

$$e^{\frac{7}{8}} < \frac{n!}{(n/e)^n\sqrt{n}} < e.$$

For many problems involving $n!$, this inequality is sufficient to give the desired result. Stirling's formula states that

$$\lim_{n\to\infty} \frac{n!}{(n/e)^n\sqrt{n}} = \sqrt{2\pi}.$$

Answer: The function $f(x)$ is a chord from the point $(m, \log m)$ to the point $(m+1, \log(m+1))$, so it is easy to see visually that $f(x) \le \log x$ for $x \in (m, m+1)$ (this is not a proof, only an explanation). The function $g(x)$ is a tangent line to $y = \log x$ at the point $(m, \log m)$, so again visually we can see that $g(x) \ge \log x$ for $m - \frac{1}{2} < x < m + \frac{1}{2}$.

Proofs of these facts go as follows: Set $F(x) = \log x - f(x)$. We see that $F'(x) = \frac{1}{x} - (\log(m+1) - \log m)$ and $F(m) = F(m+1) = 0$.

Take $t \in (m, m+1)$. We compute $F(t)$ using the Mean Value Theorem. First, $F(t) = F(t) - F(m) = F'(\xi_1)(t-m)$ for some $\xi_1 \in (m, t)$. Second, we have $-F(t) = F(m+1) - F(t) = F'(\xi_2)(m+1-t)$ for some $\xi_2 \in (t, m+1)$. Because $\xi_2 > \xi_2$, we know that $F'(\xi_1) > F'(\xi_2)$. If $F(t) = 0$, that would imply that $F'(\xi_1) = F'(\xi_2)$, which contradicts $F'(\xi_1) > F'(\xi_2)$. Moreover, because $F'(\xi_2) < F'(\xi_1)$, we can conclude that $F'(\xi_1) > 0$, showing that $F(t) > 0$. Hence, $f(t) < \log t$.

For the other inequality, start with $\log y < y - 1$. Substitute $y = \frac{x}{m}$, and we have $\log \frac{x}{m} < \frac{x}{m} - 1$. That says that $\log x < \frac{x}{m} - 1 + \log m = g(x)$.

We now know that $f(x) \le \log x \le g(x)$. Therefore, we have

$$\int_1^n f(x)\,dx \le \int_1^n \log x\,dx = n\log n - n + 1 \le \int_1^n g(x)\,dx.$$

We have

$$\begin{aligned}\int_1^n f(x)\,dx &= \sum_{m=1}^{n-1}\int_m^{m+1} f(x)\,dx = \sum_{m=1}^{n-1}\int_m^{m+1}\big((m+1-x)\log m + (x-m)\log(m+1)\big)\,dx \\ &= \sum_{m=1}^{n-1} -\frac{(m+1-x)^2}{2}\log m + \frac{(x-m)^2}{2}\log(m+1)\Bigg]_m^{m+1} \\ &= \sum_{m=1}^{n-1}\frac{1}{2}\log(m+1) + \frac{1}{2}\log m = \frac{1}{2}\log 1 + \log 2 + \log 3 + \cdots + \log(n-1) + \frac{1}{2}\log n \\ &= \log(n!) - \frac{1}{2}\log n\end{aligned}$$

Therefore,

$$\log(n!) < (n+\tfrac{1}{2})\log n - n + 1.$$

We also have

$$\begin{aligned}\int_1^n g(x)\,dx &= \int_1^{3/2} g(x)\,dx + \sum_{m=2}^{n-1}\int_{m-\frac{1}{2}}^{m+\frac{1}{2}} g(x)\,dx + \int_{n-\frac{1}{2}}^{n} g(x)\,dx \\ &= \int_1^{3/2}(x-1)\,dx + \sum_{m=2}^{n-1}\int_{m-\frac{1}{2}}^{m+\frac{1}{2}}\Big(\frac{x}{m} - 1 + \log m\Big)\,dx + \int_{n-\frac{1}{2}}^{n}\Big(\frac{x}{n} - 1 + \log n\Big)\,dx \\ &= \frac{1}{8} + \sum_{m=2}^{n-1}\left[\frac{x^2}{2m} - x + x\log m\right]_{x=m-\frac{1}{2}}^{m+\frac{1}{2}} + \left[\frac{x^2}{2n} - x + \log n\right]_{x=n-\frac{1}{2}}^{n} \\ &= \frac{1}{8} + \sum_{m=2}^{n-1}(1 - 1 + \log m) + \frac{1}{2} - \frac{1}{4n} - \frac{1}{2} + \frac{1}{2}\log n \\ &= \frac{1}{8} + \log 2 + \log 3 + \cdots + \log(n-1) + \frac{1}{2}\log n - \frac{1}{4n} \\ &= \frac{1}{8} + \log(n!) - \frac{1}{2}\log n - \frac{1}{4n} < \frac{1}{8} + \log(n!) - \frac{1}{2}\log n\end{aligned}$$

Therefore

$$\begin{aligned}n\log n - n + 1 &< \frac{1}{8} + \log(n!) - \frac{1}{2}\log n \\ \frac{7}{8} &< \log(n!) - (n+\tfrac{1}{2})\log n + n\end{aligned}$$

2. **8.7.41.** Since $e^x = \sum x^n/n!$, we know that $(e^x-1)/x = 1 + x/2! + x^2/3! + \cdots$, and hence $x/(e^x-1)$ has a power series expansion. The coefficients B_n in the expansion

$$\frac{x}{e^x-1} = \sum_{n=0}^{\infty}\frac{B_n}{n!}x^n$$

are called *Bernoulli numbers*. Prove

(a) $B_{2n+1} = 0$ for every positive integer n.

(b) We have $B_0 = 1$ and

$$\binom{n}{0}B_0 + \binom{n}{1}B_1 + \binom{n}{2}B_2 + \cdots + \binom{n}{n-1}B_{n-1} = 0 \qquad (n \geq 2)$$

(c) Show that $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$, and $B_8 = -\frac{1}{30}$.

Answer: (*a*) Set $f(x) = \frac{x}{e^x-1} + \frac{x}{2} = \frac{x+xe^x}{2(e^x-1)}$. Then

$$f(-x) = \frac{-x - xe^{-x}}{2(e^{-x}-1)} = \frac{(-x-xe^{-x})e^x}{2(e^{-x}-1)e^x} = \frac{-xe^x - x}{2(1-e^x)} = \frac{xe^x + x}{2(e^x-1)} = f(x).$$

In other words, $f(x)$ is an even function.

We also have

$$f(x) = \frac{x}{2} + \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n = 1 + \left(B_1 + \frac{1}{2}\right) x + \sum_{n=2}^{\infty} \frac{B_n}{n!} x^n.$$

Because only even powers of x show up in the Maclaurin series expansion of an even function, we conclude that $B_n = 0$ if n is a positive odd number other than 1 and that $B_1 = -\frac{1}{2}$.

(*b*) Because the power series expansion of $(e^x - 1)/x$ has constant term 1, we know immediately that the constant term in the series expansion of $x/(e^x - 1)$ is 1. For the rest, we look at the identity

$$x = (e^x - 1)\left(\sum_n \frac{B_n}{n!} x^n\right) = \left(\sum_{k>0} \frac{x^k}{k!}\right)\left(\sum_n \frac{B_n}{n!} x^n\right)$$

If $m \geq 2$, then the coefficient of x^m must be 0. But that coefficient is

$$\sum_{\substack{n+k=m \\ k\neq 0}} \left(\frac{1}{k!}\right)\left(\frac{B_n}{n!}\right) = \frac{1}{m!} \sum_{\substack{n+k=m \\ k\neq 0}} \left(\frac{m!}{k!}\right)\left(\frac{B_n}{n!}\right) = \frac{1}{m!} \sum_{\substack{n+k=m \\ k\neq 0}} \binom{m}{n} B_n$$

so

$$\sum_{n=0}^{m-1} \binom{m}{n} B_n = 0.$$

(*c*) We have already seen that $B_0 = 1$ and $B_1 = -\frac{1}{2}$. Continuing, we have

$$\begin{aligned}
0 &= \binom{3}{0} B_0 + \binom{3}{1} B_1 + \binom{3}{2} B_2 \\
0 &= 1 - \frac{3}{2} + 3B_2 \\
\frac{1}{6} &= B_2 \\
0 &= \binom{5}{0} B_0 + \binom{5}{1} B_1 + \binom{5}{2} B_2 + \binom{5}{3} B_3 + \binom{5}{4} B_4 \\
0 &= 1 - \frac{5}{2} + \frac{10}{6} + 0 + 5B_4 \\
-\frac{1}{30} &= B_4 \\
0 &= \binom{7}{0} B_0 + \binom{7}{1} B_1 + \binom{7}{2} B_2 + \binom{7}{3} B_3 + \binom{7}{4} B_4 \binom{7}{5} B_5 + \binom{7}{6} B_6 \\
0 &= 1 - \frac{7}{2} + \frac{21}{6} + 0 - \frac{35}{30} + 7B_6 \\
\frac{1}{42} &= B_6 \\
0 &= \binom{9}{0} B_0 + \binom{9}{1} B_1 + \binom{9}{2} B_2 + \binom{9}{3} B_3 + \binom{9}{4} B_4 \binom{9}{5} B_5 + \binom{9}{6} B_6 \binom{9}{7} B_7 + \binom{9}{8} B_8 \\
0 &= 1 - \frac{9}{2} + \frac{36}{6} + 0 - \frac{126}{30} + 0 + \frac{84}{42} + 0 + 9B_8 \\
-\frac{1}{30} &= B_8
\end{aligned}$$

3. **8.7.42.** If n is a nonnegative integer, the *Bernoulli polynomials* $B_n(x)$ are defined by

$$e^{xt}\frac{t}{e^t-1} = \sum_{n=0}^{\infty}\frac{(xt)^n}{n!}\cdot\sum_{n=0}^{\infty}\frac{B_n}{n!}t^n = \sum_{n=0}^{\infty}\frac{B_n(x)}{n!}t^n.$$

Prove

(a) $B_n(x)$ is a polynomial of degree n given by

$$B_n(x) = \binom{n}{0}B_0x^n + \binom{n}{1}B_1x^{n-1} + \binom{n}{2}B_2x^{n-2} + \cdots + \binom{n}{n-1}B_{n-1}x^1 + \binom{n}{n}B_n$$

Show as well that $B_n(0) = B_n$ for every nonnegative integer n, and that $B_n(1) = B_n$ for $n \geq 2$.

(b) $B'_{n+1}(x) = (n+1)B_n(x)$.

(c) $B_{n+1}(x) = B_{n+1} + (n+1)\int_0^x B_n(t)\,dt$.

(d) $\int_0^1 B_n(x)\,dx = 0$.

(e) $B_0(x) = 1$, $B_1(x) = x - \frac{1}{2}$, $B_2(x) = x^2 - x + \frac{1}{6}$, $B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$, and $B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}$.

Answer: (a) The tricky part here is using different variables in different summations. We have

$$\sum_{k=0}^{\infty}\frac{(xt)^k}{k!}\cdot\sum_{j=0}^{\infty}\frac{B_j}{j!}t^j = \sum_{n=0}^{\infty}\frac{B_n(x)}{n!}t^n.$$

Matching coefficients of t^n on both sides of the equation, we see that

$$\frac{B_n(x)}{n!} = \sum_{j+k=n}\frac{x^k}{k!}\frac{B_j}{j!}$$

Multiply by $n!$, and we have

$$B_n(x) = \sum_{j+k=n}\frac{n!}{k!j!}B_jx^k = \sum_{j=0}^{n}\binom{n}{j}B_jx^{n-j},$$

which is the desired result.

Substitution of $x = 0$ and observing that $\binom{n}{n} = 1$ gives $B_n(0) = B_n$.

For $x = 1$, we compute

$$\sum_{n=0}^{\infty}\frac{B_n(1) - B_n(0)}{n!}t^n = \frac{e^tt}{e^t-1} - \frac{t}{e^t-1} = \frac{t(e^t-1)}{e^t-1} = t.$$

Therefore, the terms on the left-hand side are 0 for $n \geq 2$, implying that $B_n(1) = B_n(0)$ for $n \geq 2$.

(b) We have

$$\frac{te^{xt}}{e^t-1} = \sum_{n=0}^{\infty}\frac{B_n(x)}{n!}t^n = 1 + \sum_{n=0}^{\infty}\frac{B_{n+1}(x)}{(n+1)!}t^{n+1}$$

$$\frac{\partial}{\partial x}\left(\frac{te^{xt}}{e^t-1}\right) = \frac{\partial}{\partial x}\left(\sum_{n=0}^{\infty}\frac{B_{n+1}(x)}{(n+1)!}t^{n+1}\right)$$

$$\frac{t^2e^{xt}}{e^t-1} = \sum_{n=0}^{\infty}\frac{B'_{n+1}(x)}{(n+1)!}t^{n+1}$$

But we also have

$$\frac{t^2e^{xt}}{e^t-1} = t\left(\frac{te^{xt}}{e^t-1}\right) = t\sum_{n=0}^{\infty}\frac{B_n(x)}{n!}t^n = \sum_{n=0}^{\infty}\frac{B_n(x)}{n!}t^{n+1}$$

Therefore,

$$\frac{B'_{n+1}(x)}{(n+1)!} = \frac{B_n(x)}{n!}$$
$$B'_{n+1}(x) = (n+1)B_n(x)$$

(c) Now the Fundamental Theorem of Calculus tells us that

$$(n+1)\int_0^x B_n(t)\,dt = B_{n+1}(x) - B_{n+1}(0) = B_{n+1}(x) - B_{n+1}.$$

(d) Now we have

$$(n+1)\int_0^1 B_n(t)\,dt = B_{n+1}(1) - B_{n+1}(0) = 0,$$

for $n+1 \geq 2$, i.e., $n > 1$. The result of course is false for $n = 0$.

(e) Now we just apply the formulas:

$$B_0(x) = \binom{0}{0}B_0x^0 = 1$$
$$B_1(x) = \binom{1}{0}B_0x^1 + \binom{1}{1}B_1x^0 = x - \frac{1}{2}$$
$$B_2(x) = \binom{2}{0}B_0x^2 + \binom{2}{1}B_1x^1 + \binom{2}{2}B_2x^0 = x^2 - x + \frac{1}{6}$$
$$B_3(x) = \binom{3}{0}B_0x^3 + \binom{3}{1}B_1x^2 + \binom{3}{2}B_2x^1 + \binom{3}{3}B_3x^0 = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$$
$$B_4(x) = \binom{4}{0}B_0x^4 + \binom{4}{1}B_1x^3 + \binom{4}{2}B_2x^2 + \binom{4}{3}B_3x^1 + \binom{4}{4}B_4x^0 = x^4 - 2x^3 + x^2 - \frac{1}{30}$$

4. **8.7.43.** Show that the following expansions are valid in the indicated intervals:

(a) $$x = \pi - 2\sum_{n=1}^{\infty}\frac{\sin nx}{n} \qquad (0 < x < 2\pi)$$

(b) $$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n\cos nx}{n^2} \qquad (-\pi \leq x \leq \pi)$$

Answer: Recall that if $f(x) \sim \sum c_k e^{ikx}$, then $c_k = (2\pi)^{-1}\int_k^{k+2\pi} f(x)e^{-ikx}\,dx$. We always compute c_0 separately.

(a) We have

$$c_0 = \frac{1}{2\pi}\int_0^{2\pi} x\,dx = \frac{4\pi^2}{2\pi\cdot 2} = \pi.$$
$$c_k = \frac{1}{2\pi}\int_0^{2\pi} xe^{-ikx}\,dx = \frac{1}{2\pi}\left[\frac{xe^{-ikx}}{-ik}\right]_0^{2\pi} + \frac{1}{2\pi}\int_0^{2\pi}\frac{e^{-ikx}}{-ik}\,dx$$
$$= \frac{1}{2\pi}\left[\frac{xe^{-ikx}}{-ik}\right]_0^{2\pi} = \frac{1}{-ik} = \frac{i}{k}$$
$$f(x) \sim \sum_{k=-\infty}^{\infty} c_ke^{ikx} = \pi + \sum_{k=0}^{\infty}\left(c_ke^{ikx} + c_{-k}e^{-ikx}\right) = \pi + \sum_{k=0}^{\infty}\left(\frac{i}{k}e^{ikx} - \frac{i}{k}e^{-ikx}\right)$$
$$= \pi + \sum_{k=0}^{\infty}\frac{i}{k}\left(\cos(kx) + i\sin(kx) - \cos(-kx) - i\sin(-kx)\right)$$
$$= \pi + \sum_{k=0}^{\infty}\frac{i}{k}\left(2i\sin(kx)\right) = \pi - \sum_{k=0}^{\infty}\frac{2\sin(kx)}{k}$$

Notice, incidentally, that Parseval's relation gives something rather interesting:

$$\begin{aligned}
\sum |c_k|^2 &= \frac{1}{2\pi}\int |f(x)|^2\,dx \\
\pi^2 + 2\sum_{k=1}^{\infty}\frac{1}{k^2} &= \frac{1}{2\pi}\int_0^{2\pi} x^2\,dx = \frac{1}{2\pi}\left(\frac{8\pi^3}{3}\right) = \frac{4\pi^2}{3} \\
2\sum_{k=1}^{\infty}\frac{1}{k^2} &= \frac{\pi^2}{3} \\
\sum_{k=1}^{\infty}\frac{1}{k^2} &= \frac{\pi^2}{6}
\end{aligned}$$

(*b*) We have

$$\begin{aligned}
c_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx = \left(\frac{1}{2\pi}\right)\frac{x^3}{3}\Bigg]_{-\pi}^{\pi} = \frac{\pi^2}{3} \\
c_k &= \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 e^{-ikx}\,dx = \frac{1}{2\pi}\left[\frac{x^2e^{-ikx}}{-ik}\right]_{-\pi}^{\pi} + \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{xe^{-ikx}}{ik}\,dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{xe^{-ikx}}{ik}\,dx = \frac{1}{\pi}\left[\frac{xe^{-ikx}}{k^2}\right]_{-\pi}^{\pi} - \frac{1}{\pi}\int_0^{\pi}\frac{e^{ikx}}{k^2}dx \\
&= \frac{1}{\pi}\left[\frac{xe^{-ikx}}{k^2}\right]_{-\pi}^{\pi} = \frac{2(-1)^k}{k^2} \\
f(x) \sim \sum_{k=-\infty}^{\infty} c_k e^{ikx} &= \frac{\pi^2}{3} + \sum_{k=0}^{\infty}\left(c_k e^{ikx} + c_{-k}e^{-ikx}\right) \\
&= \frac{\pi^2}{3} + \sum_{k=0}^{\infty}\left(\frac{2(-1)^k}{k^2}\right)\left(e^{ikx} + e^{-ikx}\right) \\
&= \frac{\pi^2}{3} + 4\sum_{k=0}^{\infty}\frac{(-1)^k\cos kx}{k^2}
\end{aligned}$$

Again, Parseval's relation gives an interesting result:

$$\begin{aligned}
\frac{\pi^4}{9} + \sum_{k=1}^{\infty}\frac{8}{k^4} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} x^4\,dx = \frac{1}{2\pi}\left(\frac{2\pi^5}{5}\right) = \frac{\pi^4}{5} \\
\sum_{k=1}^{\infty}\frac{8}{k^4} &= \frac{4\pi^4}{45} \\
\sum_{k=1}^{\infty}\frac{1}{k^4} &= \frac{\pi^4}{90}
\end{aligned}$$

5. **10.1.2.** Show that $\Gamma(1) = 1$ and that $\Gamma(x+1) = x\Gamma(x)$ for all $x > 0$. Deduce that $\Gamma(n+1) = n!$ for every nonnegative integer n.

Answer: We have $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt$, and therefore $\Gamma(1) = \int_0^\infty e^{-t}\,dt = \lim_{y\to\infty} -e^{-t}\Big]_0^y = \lim_{y\to\infty}(-e^{-y}) + 1 = 1$. Setting $u = t^x$, $dv = e^{-t}\,dt$, $du = tx^{t-1}\,dt$, $v = -e^{-t}$, and integrating by parts gives

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t}\,dt = \Big[t^x(-e^{-t})\Big]_{t=0}^{\infty} + x\int_0^\infty t^{x-1}e^{-t}\,dt = x\Gamma(x),$$

where we have used the fact that $\lim_{t\to\infty} t^x/e^t = 0$ for any x. Induction now gives $\Gamma(n+1) = n!$.