Mathematics 805
Homework 8
Due Friday, March 27, 1 PM

1. A good reference for the history of factorials and $\Gamma(x)$ is an article by P.J. Davis in *The American Mathematical Monthly*, vol. 66, 1959, pp. 849–869, available through `www.jstor.org`.

This proof of a less precise form of Stirling's formula is taken from Rudin's *Principles of Mathematical Analysis*.

For m a positive integer, define

$$\begin{aligned} f(x) &= (m+1-x)\log m + (x-m)\log(m+1) && m \le x \le m+1 \\ g(x) &= \frac{x}{m} - 1 + \log m && m - \tfrac{1}{2} \le x < m + \tfrac{1}{2} \end{aligned}$$

You might find it helpful to graph $f(x)$ and $g(x)$ to see why one might define such functions.

Prove that

$$f(x) \le \log x \le g(x).$$

This shows that if n is a positive integer,

$$\int_1^n f(x)\,dx \le \int_1^n \log x\,dx \le \int_1^n g(x)\,dx.$$

Use this inequality to show that

$$\tfrac{7}{8} < \log(n!) - (n + \tfrac{1}{2})\log n + n < 1,$$

thereby proving that

$$e^{\frac{7}{8}} < \frac{n!}{(n/e)^n \sqrt{n}} < e.$$

For many problems involving $n!$, this inequality is sufficient to give the desired result. Stirling's formula states that

$$\lim_{n\to\infty} \frac{n!}{(n/e)^n \sqrt{n}} = \sqrt{2\pi}.$$