Mathematics 805
Homework 9
Due Friday, April 3, 1 PM

1. As before, let $B_n(x)$ be the Bernoulli polynomial of degree n.
   (*a*) Show that $B_n(x+1) - B_n(x) = nx^{n-1}$.
   (*b*) By using part (*a*), derive a formula expressing

$$\sum_{j=1}^{k} j^r$$

in terms of $B_{r+1}(x)$.

Answer: (*a*) We have

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{B_n(x+1) - B_n(x)}{n!} t^n &= \frac{(e^{t(x+1)} - e^{tx})t}{e^t - 1} = \frac{(e^t - 1)e^{tx}t}{e^t - 1} \\ &= te^{tx} = t\sum_{n=0}^{\infty} \frac{(tx)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} t^{n+1} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} t^n\end{aligned}$$

Equating coefficients of t^n, we get

$$\begin{aligned}\frac{B_n(x+1) - B_n(x)}{n!} &= \frac{x^{n-1}}{(n-1)!}. \\ \frac{B_n(x+1) - B_n(x)}{n} &= x^{n-1}\end{aligned}$$

Notice incidentally that there is no problem with $n = 0$, because $B_0(x+1) = B_0(x) = 1$, so there is no $n = 0$ term on the left-hand side of the initial equation.

(*b*) We have

$$\frac{B_{r+1}(x+1) - B_{r+1}(x)}{r+1} = x^r,$$

so

$$\sum_{j=1}^{k} j^r = \frac{1}{r+1} \sum_{j=1}^{k} \left[B_{r+1}(j+1) - B_{r+1}(j)\right] = \frac{B_{r+1}(k+1) - B_{r+1}(1)}{r+1}.$$

Because $B_{r+1}(1) = B_{r+1}(0) = B_{r+1}$, the above formula is sometimes written as $(B_{r+1}(k+1) - B_{r+1})/(r+1)$. A quick check: $1^3 + 2^3 + \cdots + 10^3 = 3025$. We computed last week that $B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}$, and indeed $(B_4(11) - B_4)/4 = 3025$.

2. **8.7.44.** Obtain the following Fourier expansions for the Bernoulli polynomials on the interval $[0, 1]$:

(*a*) $$B_{2n}(x) = (-1)^{n+1} \frac{2(2n)!}{(2\pi)^{2n}} \sum_{k=1}^{\infty} \frac{\cos 2k\pi x}{k^{2n}} \qquad n = 1, 2, \ldots$$

(*b*) $$B_{2n+1}(x) = (-1)^{n+1} \frac{2(2n+1)!}{(2\pi)^{2n+1}} \sum_{k=1}^{\infty} \frac{\sin 2k\pi x}{k^{2n+1}} \qquad n = 1, 2, \ldots$$

Answer: Start with

$$z^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n \cos nz}{n^2}$$

valid for $-\pi \leq z \leq \pi$. Let $z = y - \pi$. Note that $\cos nz = \cos(ny - n\pi) = (\cos ny)(\cos n\pi) = (-1)^n \cos ny$, so we have

$$\begin{aligned} y^2 - 2\pi y + \pi^2 = (y-\pi)^2 &= \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n \cos ny(-1)^n}{n^2} = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{\cos ny}{n^2} \\ y^2 - 2\pi y + \frac{2\pi^2}{3} &= 4\sum_{n=1}^{\infty} \frac{\cos ny}{n^2} \end{aligned}$$

valid for $0 \leq y \leq 2\pi$. Now let $y = 2\pi x$, and we have

$$\begin{aligned} (2\pi x)^2 - 2\pi(2\pi x) + \frac{2\pi^2}{3} &= 4\sum_{n=1}^{\infty} \frac{\cos 2\pi nx}{n^2} \\ 4\pi^2 x^2 - 4\pi^2 x + \frac{2\pi^2}{3} &= 4\sum_{n=1}^{\infty} \frac{\cos 2\pi nx}{n^2} \\ B_2(x) = x^2 - x + \frac{1}{6} &= \frac{1}{\pi^2}\sum_{n=1}^{\infty} \frac{\cos 2\pi nx}{n^2} \end{aligned}$$

valid for $0 \leq x \leq 1$. This is the case $n = 1$ of formula (a), which we use to start the induction.

Assume now that formula (a) is true for n. We will show that formula (b) is also true for n, using term-by-term integration. To see that the Fourier series is uniformly convergent, apply the Weierstrass M-test using the comparison series $\sum k^{-2n}$. We have

$$\begin{aligned} B_{2n+1}(x) &= B_{2n+1} + (2n+1)\int_0^x B_{2n}(y)\,dy = (2n+1)\int_0^x B_{2n}(x)\,dx = \\ &= (2n+1)(-1)^{n+1}\frac{2(2n)!}{(2\pi)^{2n}}\sum_{k=1}^{\infty}\int_0^x \frac{\cos 2k\pi y}{k^{2n}}\,dy \\ &= (-1)^{n+1}\frac{2(2n+1)!}{(2\pi)^{2n}}\sum_{k=1}^{\infty}\frac{\sin 2k\pi y}{(2k\pi)k^{2n}}\Bigg]_0^x = (-1)^{n+1}\frac{2(2n+1)!}{(2\pi)^{2n+1}}\sum_{k=1}^{\infty}\frac{\sin 2k\pi x}{k^{2n+1}} \end{aligned}$$

Now we assume that formula (b) is true for n, and show that formula (a) is true for $n+1$. This is trickier, because $B_{2n+2} \neq 0$ and $\cos 0 \neq 0$, so there are constants of integration to be dealt with. Again, we can integrate term-by-term, because we can apply the Weierstrass M-test using the comparison series $\sum k^{-2n-1}$. We have

$$\begin{aligned} B_{2n+2}(x) &= B_{2n+2} + (2n+2)\int_0^x B_{2n+1}(y)\,dy \\ &= B_{2n+2} + (2n+2)(-1)^{n+1}\frac{2(2n+1)!}{(2\pi)^{2n+1}}\sum_{k=1}^{\infty}\int_0^x \frac{\sin 2k\pi y}{k^{2n+1}}\,dy \\ &= B_{2n+2} + (-1)^{n+1}\frac{2(2n+2)!}{(2\pi)^{2n+1}}\sum_{k=1}^{\infty}\frac{-\cos 2k\pi y}{(2k\pi)k^{2n+1}}\Bigg]_0^x \\ &= \left(B_{2n+2} + (-1)^{n+1}\frac{2(2n+2)!}{(2\pi)^{2n+2}}\sum_{k=1}^{\infty}\frac{1}{k^{2n+2}}\right) + (-1)^{n+2}\frac{2(2n+2)!}{(2\pi)^{2n+2}}\sum_{k=1}^{\infty}\frac{\cos 2k\pi x}{k^{2n+2}} \end{aligned}$$

We need to show that the constant term in parentheses is 0. Call it C_{2n+2}, so we have

$$B_{2n+2}(x) = C_{2n+2} + (-1)^{n+2}\frac{2(2n+2)!}{(2\pi)^{2n+2}}\sum_{k=1}^{\infty}\frac{\cos 2k\pi x}{k^{2n+2}}$$

The only way that I can see to show that C_{2n+2} is 0 is to integrate this equation from 0 to 1 with respect to x. We know that $\int_0^1 B_k(x)\,dx = 0$ for $k = 1, 2, \ldots$. We know that $\int_0^1 \cos 2k\pi x\,dx = 0$ for $k = 1, 2, \ldots$. Term-by-term integration is again justifiable because we have a uniformly convergent series using the Weierstrass M-test. The conclusion is that $\int_0^1 C_{2n+2}\,dx = 0$, which means that $C_{2n+2} = 0$; hence,

$$B_{2n+2}(x) = (-1)^{n+2}\frac{2(2n+2)!}{(2\pi)^{2n+2}}\sum_{k=1}^{\infty}\frac{\cos 2k\pi x}{k^{2n+2}}$$

establishing the induction.

3. **8.7.45.** Recall that the series $\sum_{n=1}^{\infty} n^{-s}$ is convergent for $s > 1$ and divergent for $s \le 1$. The Riemann zeta-function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s} \qquad (\forall s > 1)$$

(a) Show that

$$\zeta(2n) = (-1)^{n+1}\frac{(2\pi)^{2n}B_{2n}}{2(2n)!} \qquad n = 1, 2, \ldots$$

(b) Show that $\zeta(6) = \frac{\pi^6}{945}$ and $\zeta(8) = \frac{\pi^8}{9450}$.

Answer: (a) Take formula (a) from the previous problem, and substitute $x = 0$:

$$B_{2n}(x) = (-1)^{n+1}\frac{2(2n)!}{(2\pi)^{2n}}\sum_{k=1}^{\infty}\frac{\cos 2k\pi x}{k^{2n}}$$

$$B_{2n} = B_{2n}(0) = (-1)^{n+1}\frac{2(2n)!}{(2\pi)^{2n}}\sum_{k=1}^{\infty}\frac{1}{k^{2n}} = (-1)^{n+1}\frac{2(2n)!}{(2\pi)^{2n}}\zeta(2n)$$

$$(-1)^{n+1}\frac{B_{2n}(2\pi)^{2n}}{2(2n)!} = \zeta(2n)$$

(b) We know that $B_6 = \frac{1}{42}$, so $\zeta(6) = \frac{1}{42}\cdot\frac{(2\pi)^6}{2\cdot 6!} = \frac{1}{42}\cdot\frac{64\pi^6}{1440} = \frac{\pi^6}{945}$. As a quick check, we can compute that $\frac{\pi^6}{945} \approx 1.01734306$, while $\sum_{k=1}^{5} k^{-6} \approx 1.01730488$.

We also have $B_8 = -\frac{1}{30}$, so $\zeta(8) = \frac{1}{30}\cdot\frac{(2\pi)^8}{2\cdot 8!} = \frac{256\pi^8}{30\cdot 80640} = \frac{\pi^8}{9450}$. Again, we can compute that $\frac{\pi^8}{9450} \approx 1.00407736$, while $\sum_{k=1}^{5} k^{-8} \approx 1.00407648$.

4. **8.7.46.** (a) Show that if $\alpha \notin \mathbf{Z}$, then

$$\cos\alpha x = \frac{\sin\alpha\pi}{\pi}\left(\frac{1}{\alpha} - \frac{2\alpha}{\alpha^2 - 1^2}\cos x + \frac{2\alpha}{\alpha^2 - 2^2}\cos 2x - \cdots\right)$$

and deduce that

$$\frac{\pi}{\sin\alpha\pi} = \frac{1}{\alpha} - \frac{2\alpha}{\alpha^2 - 1^2} + \frac{2\alpha}{\alpha^2 - 2^2} - \cdots$$

(b) Plugging in $x = 0$ and $x = \pi$ in the above series for $\cos\alpha x$ and relabeling, prove

(i)
$$\csc\pi x = \frac{1}{\pi x} + \frac{2x}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{x^2 - n^2}$$

(ii)
$$\cot \pi x = \frac{1}{\pi x} + \frac{2x}{\pi}\sum_{n=1}^{\infty} \frac{1}{x^2 - n^2}$$

Answer: We begin by expanding $\cos \alpha x$ in a Fourier series from $-\pi$ to π. Because the function is an even one, we know that the result will only have terms involving $\cos nx$. We will work in terms of exponential functions, using the fact that $\cos x = \frac{e^{ix}+e^{-ix}}{2}$. As usual, we must compute c_0 separately.

We have

$$\begin{aligned}
c_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\alpha x}+e^{-i\alpha x}}{2}\,dx = \frac{1}{2\pi}\left[\frac{e^{i\alpha x}-e^{-i\alpha x}}{2i\alpha}\right]_{-\pi}^{\pi} = \frac{1}{2\pi\alpha}(\sin\alpha\pi - \sin(-\alpha\pi)) = \frac{\sin\alpha\pi}{\pi\alpha}.\\
c_k &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\alpha x}+e^{-i\alpha x}}{2} e^{-ikx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{ix(\alpha-k)}+e^{-ix(\alpha+k)}}{2}\,dx = \frac{1}{2\pi}\left[\frac{e^{ix(\alpha-k)}}{2i(\alpha-k)} - \frac{e^{-ix(\alpha+k)}}{2i(\alpha+k)}\right]_{-\pi}^{\pi}\\
&= \frac{1}{2\pi}\left(\frac{e^{i\pi(\alpha-k)}-e^{-i\pi(\alpha-k)}}{2i(\alpha-k)} + \frac{e^{i\pi(\alpha+k)}-e^{-i\pi(\alpha+k)}}{2i(\alpha+k)}\right) = \frac{1}{2\pi}\left(\frac{\sin(\alpha\pi-k\pi)}{\alpha-k} + \frac{\sin(\alpha\pi+k\pi)}{\alpha+k}\right)\\
&= \frac{1}{2\pi}\left(\frac{(-1)^k\sin(\alpha\pi)}{\alpha-k} + \frac{(-1)^k\sin(\alpha\pi)}{\alpha+k}\right) = \frac{(-1)^k\sin\alpha\pi}{2\pi}\left(\frac{1}{\alpha-k}+\frac{1}{\alpha+k}\right) = \frac{(-1)^k\alpha\sin\alpha\pi}{\pi(\alpha^2-k^2)}
\end{aligned}$$

Notice that $c_k = c_{-k}$. Therefore

$$\begin{aligned}
\cos\alpha x &\sim \sum_{k=-\infty}^{\infty} c_k e^{ikx} = \frac{\sin\alpha\pi}{\pi\alpha} + \sum_{k=1}^{\infty} c_k e^{ikx} + c_{-k}e^{-ikx} = \frac{\sin\alpha\pi}{\pi\alpha} + \sum_{k=1}^{\infty}\frac{(-1)^k\alpha\sin\alpha\pi}{\pi(\alpha^2-k^2)}(e^{ikx}+e^{-ikx})\\
&= \frac{\sin\alpha\pi}{\pi\alpha} + \sum_{k=1}^{\infty}\frac{(-1)^k 2\alpha\sin\alpha\pi}{\pi(\alpha^2-k^2)}\left(\frac{e^{ikx}+e^{-ikx}}{2}\right) = \frac{\sin\alpha\pi}{\pi\alpha} + \sum_{k=1}^{\infty}\frac{(-1)^k 2\alpha\sin\alpha\pi}{\pi(\alpha^2-k^2)}\cos kx\\
&= \frac{\sin\alpha\pi}{\pi}\left(\frac{1}{\alpha} + \sum_{k=1}^{\infty}\frac{(-1)^k 2\alpha}{\alpha^2-k^2}\cos kx\right) = \frac{\sin\alpha\pi}{\pi}\left(\frac{1}{\alpha} - \frac{2\alpha\cos x}{\alpha^2-1} + \frac{2\alpha\cos 2x}{\alpha^2-4}\cos 2x - \cdots\right)
\end{aligned}$$

The Fourier series converges to $\cos\alpha x$ for $x \in (-\pi,\pi)$ because $\cos\alpha x$ is continuously differentiable. The series also converges at $\pm\pi$ because $\cos\alpha\pi = \cos(-\alpha\pi)$, and the left- and right-hand derivatives exist at $\pm\pi$.

Substituting $x = 0$, we have

$$\begin{aligned}
1 &= \frac{\sin\alpha\pi}{\pi}\left(\frac{1}{\alpha} + \sum_{k=1}^{\infty}\frac{(-1)^k 2\alpha}{\alpha^2-k^2}\right)\\
\frac{\pi}{\sin\alpha\pi} &= \frac{1}{\alpha} + \sum_{k=1}^{\infty}\frac{(-1)^k 2\alpha}{\alpha^2-k^2}
\end{aligned}$$

(*b*) Take the previous equation and replace α by x to get

$$\begin{aligned}
\frac{\pi}{\sin\pi x} &= \frac{1}{x} + \sum_{k=1}^{\infty}\frac{(-1)^k 2x}{x^2-k^2}\\
\csc\pi x &= \frac{1}{\pi x} + \sum_{k=1}^{\infty}\frac{(-1)^k 2x}{\pi(x^2-k^2)} = \frac{1}{\pi x} + \frac{2x}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{x^2-k^2}
\end{aligned}$$

We can also take the Fourier series and substitute $x = \pi$, getting

$$\begin{aligned}
\cos\alpha\pi &= \frac{\sin\alpha\pi}{\pi}\left(\frac{1}{\alpha} + \sum_{k=1}^{\infty}\frac{(-1)^k 2\alpha}{\alpha^2-k^2}\cos k\pi\right) = \frac{\sin\alpha\pi}{\pi}\left(\frac{1}{\alpha} + \sum_{k=1}^{\infty}\frac{2\alpha}{\alpha^2-k^2}\right)\\
\cot\alpha\pi &= \frac{1}{\pi}\left(\frac{1}{\alpha} + \sum_{k=1}^{\infty}\frac{2\alpha}{\alpha^2-k^2}\right) = \frac{1}{\pi\alpha} + \frac{2\alpha}{\pi}\sum_{k=1}^{\infty}\frac{1}{\alpha^2-k^2}
\end{aligned}$$

Relabeling α as x gives the desired result.

5. **10.7.1.** Evaluate each improper integral.

$$(a) \quad \int_0^{\frac{\pi}{2}} \sqrt{\sin x \tan x}\, dx \qquad\qquad (b) \quad \int_{-1}^{1} \frac{dx}{\sqrt[3]{x}}$$

$$(c) \quad \int_0^{\frac{\pi}{2}} x \cot x\, dx \qquad\qquad (d) \quad \int_0^{1} \frac{\log x}{\sqrt{x}}\, dx$$

Answer: (a) We have

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sqrt{\sin x \tan x}\, dx &= \lim_{t\to\frac{\pi}{2}} \int_0^{t} \sqrt{\sin x \tan x}\, dx = \lim_{t\to\frac{\pi}{2}} \int_0^{t} \sqrt{\frac{\sin^2 x}{\cos x}}\, dx = \lim_{t\to\frac{\pi}{2}} \int_0^{t} \frac{\sin x}{\sqrt{\cos x}}\, dx \\ &= \lim_{t\to\frac{\pi}{2}} \int_{x=0}^{x=t} \frac{-du}{\sqrt{u}} = \lim_{t\to\frac{\pi}{2}} -2\sqrt{u}\Big]_{x=0}^{x=t} = \lim_{t\to\frac{\pi}{2}} -2\sqrt{\cos x}\Big]_{x=0}^{x=t} = 2.\end{aligned}$$

(b) We have

$$\int_{-1}^{1} \frac{dx}{\sqrt[3]{x}} = \lim_{t\to 0} \int_{-1}^{0} \frac{dx}{\sqrt[3]{x}} + \lim_{t\to 0} \int_{0}^{1} \frac{dx}{\sqrt[3]{x}} = \lim_{t\to 0} \frac{3}{2} x^{2/3}\Big]_{-1}^{t} + \lim_{t\to 0} \frac{3}{2} x^{2/3}\Big]_{t}^{1} = 0.$$

(c) Note that this is not really an improper integral, because $\lim\limits_{x\to 0} x \cot x = 1$. However, it is a tricky integral to do:

$$\begin{aligned}\int_0^{\frac{\pi}{2}} x \cot x\, dx &= \int_0^{\frac{\pi}{4}} x \cot x\, dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x \cot x\, dx = \int_0^{\frac{\pi}{4}} x \cot x\, dx + \int_0^{\frac{\pi}{4}} \left(\frac{\pi}{2} - y\right) \cot\left(\frac{\pi}{2} - y\right)\, dy \\ &= \int_0^{\frac{\pi}{4}} x \cot x\, dx + \int_0^{\frac{\pi}{4}} \left(\frac{\pi}{2} - y\right) \tan y\, dy = \int_0^{\frac{\pi}{4}} x \cot x\, dx + \int_0^{\frac{\pi}{4}} \frac{\pi}{2} \tan y\, dy - \int_0^{\frac{\pi}{4}} y \tan y\, dy \\ &= \int_0^{\frac{\pi}{4}} x \cot x\, dx - \int_0^{\frac{\pi}{4}} x \tan x\, dy - \left[\frac{\pi}{2} \log(\cos y)\right]_0^{\frac{\pi}{4}} = \int_0^{\frac{\pi}{4}} x(\cot x - \tan x)\, dx + \frac{\pi}{4} \log 2 \\ &= \frac{\pi}{4} \log 2 + \int_0^{\frac{\pi}{4}} x \left(\frac{\cos x}{\sin x} - \frac{\sin x}{\cos x}\right)\, dx = \frac{\pi}{4} \log 2 + \int_0^{\frac{\pi}{4}} x \left(\frac{\cos^2 x - \sin^2 x}{\cos x \sin x}\right)\, dx \\ &= \frac{\pi}{4} \log 2 + \int_0^{\frac{\pi}{4}} 2x \left(\frac{\cos 2x}{\sin 2x}\right)\, dx = \frac{\pi}{4} \log 2 + \int_0^{\frac{\pi}{4}} 2x \cot 2x\, dx = \frac{\pi}{4} \log 2 + \frac{1}{2} \int_0^{\frac{\pi}{2}} z \cot z\, dz\end{aligned}$$

$$\begin{aligned}\frac{1}{2} \int_0^{\frac{\pi}{2}} z \cot z\, dz &= \frac{\pi}{4} \log 2 \\ \int_0^{\frac{\pi}{2}} z \cot z\, dz &= \frac{\pi}{2} \log 2\end{aligned}$$

(d) We have

$$\begin{aligned}\int_0^{1} \frac{\log x}{\sqrt{x}}\, dx &= \lim_{t\to 0} \int_t^{1} \frac{\log x}{\sqrt{x}}\, dx = \lim_{t\to 0} \int_{x=t}^{x=1} \frac{\log u^2}{u} 2u\, du = 2 \lim_{t\to 0} \int_{x=t}^{x=1} \log u^2\, du \\ &= 4 \lim_{t\to 0} \int_{x=t}^{x=1} \log u\, du = 4 \lim_{t\to 0} \left[u \log u - u\right]_{x=t}^{x=1} = 4 \lim_{t\to 0} \left[\sqrt{x} \log \sqrt{x} - \sqrt{x}\right]_{x=t}^{x=1} = -4.\end{aligned}$$