

Mathematics 805  
Homework 10  
Due Friday, April 17, 1 PM

1. As before, let  $B_n(x)$  be the Bernoulli polynomial of degree  $n$ . Show that  $B_n(1-x) = (-1)^n B_n(x)$ . What can you conclude about  $B_{2n+1}(\frac{1}{2})$ ?

*Answer:* Using the definition of  $B_n(x)$ , we have

$$\sum_{n=0}^{\infty} \frac{B_n(1-x)}{n!} t^n = \frac{te^{(1-x)t}}{e^t - 1} = \frac{te^{(1-x)t}}{e^t - 1} \cdot \frac{e^{-t}}{e^{-t}} = \frac{te^{-xt}}{1 - e^{-t}} = \frac{(-t)e^{x(-t)}}{e^{-t} - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} (-t)^n$$

Comparing coefficients, we see that  $B_n(1-x) = B_n(x)(-1)^n$ , as desired. Notice that this shows that  $B_{2n+1}(\frac{1}{2}) = (-1)B_{2n+1}(\frac{1}{2})$ , meaning that  $B_{2n+1}(\frac{1}{2}) = 0$ .

2. Start with the formula

$$\cot \pi x = \frac{1}{\pi x} + \frac{2x}{\pi} \sum_{n=1}^{\infty} \frac{1}{x^2 - n^2}$$

integrate term-by-term, exponentiate, and do a bit of algebra to get an infinite product formula for  $\frac{\sin x}{x}$ . Justifying the validity of integrating term-by-term is not easy, so you can omit the justification for once.

*Answer:* Let  $\pi x = y$ , so that  $x = y/\pi$ . The formula becomes

$$\cot y - \frac{1}{y} = \frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{2y}{(y/\pi)^2 - n^2} = \sum_{n=1}^{\infty} \frac{2y}{y^2 - \pi^2 n^2}.$$

The left-hand side of the equation simplifies to  $(y \cos y - \sin y)/y \sin y$ . The Taylor expansion of the numerator starts  $(y - \frac{y^3}{2} + \dots) - (y - \frac{y^3}{6} + \dots)$ , so the leading term in the numerator is  $-\frac{1}{3}y^3$ . The Taylor expansion of the denominator starts  $y^2 - \frac{y^4}{6} + \dots$ . We can therefore cancel  $y^2$  from both the numerator and denominator, and the result will be a quotient of Taylor series, with a denominator starting  $1 + \dots$ . A theorem assures us that such a function is analytic in a neighborhood of 0, and hence can certainly be integrated. It is easy to check that the indefinite integral of  $\cot y - \frac{1}{y}$  is  $\log(\frac{\sin y}{y})$ , extended by continuity to be 0 when  $y \rightarrow 0$ .

The right-hand side of the equation is uniformly convergent for  $y \in [-\frac{1}{2}, \frac{1}{2}]$  (for example), using the Weierstrass  $M$ -test and comparing with  $\sum \frac{1}{\pi^2 n^2 - \frac{1}{4}}$ . So we can integrate the right-hand side of the equation term-by-term also. Assuming that  $|z| < \frac{1}{2}$ , we have

$$\begin{aligned} \int_0^z \left( \cot y - \frac{1}{y} \right) dy &= \int_0^z \left( \sum_{n=1}^{\infty} \frac{2y}{y^2 - n^2 \pi^2} \right) dy = \sum_{n=1}^{\infty} \left( \int_0^z \frac{2y}{y^2 - n^2 \pi^2} dy \right) \\ \log \left| \frac{\sin y}{y} \right| \Big|_{y=0}^z &= \sum_{n=1}^{\infty} \log |y^2 - n^2 \pi^2| \Big|_{y=0}^z \\ \log \frac{\sin z}{z} &= \sum_{n=1}^{\infty} \log (n^2 \pi^2 - z^2) - \log (n^2 \pi^2) = \sum_{n=1}^{\infty} \log \left( \frac{n^2 \pi^2 - z^2}{n^2 \pi^2} \right) \\ &= \sum_{n=1}^{\infty} \log \left( 1 - \frac{z^2}{n^2 \pi^2} \right) = \log \prod_{n=1}^{\infty} \left( 1 - \frac{z^2}{n^2 \pi^2} \right) \\ \frac{\sin z}{z} &= \prod_{n=1}^{\infty} \left( 1 - \frac{z^2}{n^2 \pi^2} \right) \end{aligned}$$

The amazing fact is that this product, which we can only derive in some neighborhood of 0 (and surely not for  $z > \pi$ , at which point the original expansion was not valid), is in fact convergent and valid for all values of  $z$ .

3. **8.4.1.** (1) Give  $a < b$ , define the function

$$f(x, a, b) = \begin{cases} \exp\left[\frac{-1}{(x-a)(b-x)}\right] & \text{if } x \in (a, b) \\ 0 & \text{if } x \notin (a, b) \end{cases}.$$

Show that  $f \in C^\infty(\mathbf{R})$ .

(2) Given  $\delta \in (0, 1)$ , let  $g(x) = f(x, -1, -1 + \delta)$  with  $f = f(x, a, b)$  defined above. Define

$$h(x) = \frac{1}{A} \int_{-1}^x g(t) dt,$$

where  $A = \int_{-1}^{-1+\delta} g(t) dt$ . Show that  $h \in C^\infty(\mathbf{R})$  and  $h(x) \equiv 1$  for all  $x \geq -1 + \delta$ , while  $h(x) \equiv 0$  for all  $x \leq -1$ .

(3) Let  $\phi(x) = h(-|x|)$  for all  $x \in \mathbf{R}$ . Show that  $\phi \in C^\infty(\mathbf{R})$ ,  $\phi(x) \equiv 1$  for all  $x \in [-1 + \delta, 1 - \delta]$ , and  $\phi(x) \equiv 0$  for all  $|x| > 1$ .

*Answer:* (1) For notational simplicity, we write  $f(x)$  for  $f(x, a, b)$  and  $\exp(x)$  for  $e^x$ . The function  $f(x)$  is clearly  $C^\infty$  except perhaps at  $x = a$  and  $x = b$ . We need to show that  $f^{(n)}(a) = f^{(n)}(b) = 0$  for all  $n \geq 0$ . It suffices to do this for  $x = a$ , because the same arguments apply for  $x = b$ .

To begin, we must compute  $\lim_{x \rightarrow a^+} \exp(-1/(x-a))$ . Let  $z = \frac{1}{x-a}$ , and then  $\lim_{x \rightarrow a^+} \exp(-1/(x-a)) = \lim_{z \rightarrow \infty} \exp(-z) = 0$ . This shows that  $f(x)$  is continuous at  $x = a$ .

In general, we need to show that  $\lim_{x \rightarrow a^+} \frac{f^{(n)}(x) - 0}{x-a} = 0$ . To do this, we start by proving using induction that  $f^{(n)}(x) = \frac{p_n(x)f(x)}{(x-a)^k(x-b)^k}$ , where  $p_n(x)$  is a polynomial and  $k$  is some positive integer. For the case  $n = 1$ , we have  $f(x) = \exp(\frac{-1}{(x-a)(b-x)})$ , and so  $\log(f(x)) = \frac{-1}{(x-a)(b-x)} = (x-a)^{-1}(x-b)^{-1}$ . Differentiation yields  $\frac{f'(x)}{f(x)} = -(x-a)^{-2}(x-b)^{-1} - (x-a)^{-1}(x-b)^{-2} = \frac{p_1(x)}{(x-a)^2(x-b)^2}$ .

Assuming that we have established  $f^{(n)}(x) = \frac{p_n(x)f(x)}{(x-a)^k(x-b)^k}$ , we take logarithms to get  $\log(f^{(n)}(x)) = \log(p_n(x)) + \log(f(x)) - k \log(x-a) - k \log(x-b)$ . Differentiation yields

$$\begin{aligned} \frac{f^{(n+1)}(x)}{f^{(n)}(x)} &= \frac{p'_n(x)}{p_n(x)} + \frac{p_1(x)}{(x-a)^2(x-b)^2} - \frac{k}{x-a} - \frac{k}{x-b} \\ f^{(n+1)}(x) &= \frac{f(x)}{(x-a)^k(x-b)^k} \left( p'_n(x) + \frac{p_1(x)p_n(x)}{(x-a)^2(x-b)^2} - \frac{kp_n(x)}{x-a} - \frac{kp_n(x)}{x-b} \right) \\ &= \frac{f(x)p_{n+1}(x)}{(x-a)^j(x-b)^j}, \end{aligned}$$

establishing the induction.

The rest is much simpler. Using the substitution  $z = \frac{1}{x-a}$ , we have

$$\begin{aligned} f_+^{(n)}(x) &= \lim_{x \rightarrow a^+} \frac{f^{(n-1)}(x) - 0}{x-a} = \lim_{x \rightarrow a^+} \frac{\frac{p_{n-1}(x)f(x)}{(x-a)^k(x-b)^k}}{x-a} = \frac{p_{n-1}(a)}{(a-b)^k} \lim_{x \rightarrow a^+} \frac{f(x)}{(x-a)^{k+1}} \\ &= \frac{p_{n-1}(a)}{(a-b)^k} \lim_{x \rightarrow a^+} \frac{\exp(\frac{-1}{(x-a)(b-x)})}{(x-a)^{k+1}} = \frac{p_{n-1}(a)}{(a-b)^k} \lim_{z \rightarrow \infty} z^{k+1} \exp(-z) \\ &= \frac{p_{n-1}(a)}{(a-b)^k} \lim_{z \rightarrow \infty} \frac{z^{k+1}}{\exp(z)} = 0 \end{aligned}$$

(2) The integral of a  $C^\infty$  function is  $C^\infty$ , so  $h \in C^\infty$ . The other properties are immediate from the definition of  $g(x)$ .

(3) Because the function  $|x|$  is not differentiable at 0, we need to be a bit concerned about the value  $x = 0$  when showing that  $\phi(x)$  is  $C^\infty$ . But if  $|x| < 1 - \delta$ , we have  $\phi(x)$  is identically 1, and hence is  $C^\infty$  for in that neighborhood of 0. For other values of  $x$ ,  $\phi(x)$  is a composition of  $C^\infty$  functions, and hence is  $C^\infty$ . The other properties are immediate.

4. **8.7.38.** Recall that, for any  $\alpha \in \mathbf{R}$ , we have Newton's Binomial Series

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n \quad (|x| < 1).$$

For  $\alpha$  a nonnegative integer, the series terminates and we get the binomial formula.

(a) Show that  $\sum_{n=0}^{\infty} \binom{\alpha}{n}$  converges for  $\alpha > -1$  and diverges for  $\alpha \leq -1$ .

(b) Show that  $\sum_{n=0}^{\infty} (-1)^n \binom{\alpha}{n}$  converges for  $\alpha \geq 0$  and diverges for  $\alpha < 0$ .

(c) Show that if  $\alpha \geq 0$ , then the binomial series converges uniformly to  $(1+x)^\alpha$  on  $[-1, 1]$ , and, when  $\alpha > -1$ , it converges uniformly to  $(1+x)^\alpha$  uniformly on  $[-1 + \delta, 1]$  for  $\delta > 0$ . In particular, we have  $\sum_{n=0}^{\infty} \binom{\alpha}{n} = 2^\alpha$  for all  $\alpha > -1$  and  $\sum_{n=0}^{\infty} (-1)^n \binom{\alpha}{n} = 0$  for all  $\alpha > 0$ .

*Answer:* It's actually simplest to answer the three parts of the question all at once. We may assume in all that follows that  $\alpha$  is neither 0 nor a positive integer.

Note first that if  $n > \alpha$ , then  $\binom{\alpha}{n}$  and  $\binom{\alpha}{n+1}$  will have opposite signs, because  $\binom{\alpha}{n+1}/\binom{\alpha}{n} = \frac{\alpha-n}{n+1} < 0$ . Therefore,  $\sum (-1)^n \binom{\alpha}{n}$  is eventually a series of terms of the same sign, so we can apply the standard tests to see if this series converges.

Unfortunately, we know that the ratio test is useless here. One useful test is **Raabe's Test** (p. 62). This test states that  $\sum x_n$  converges if  $x_{n+1}/x_n \leq 1 - r/n$  is ultimately true for some  $r > 1$ , and  $\sum x_n$  diverges if  $x_{n+1}/x_n \geq 1 - 1/n$  is ultimately true.

The series we study is  $\sum |\binom{\alpha}{n-1}|$ , where we have reindexed to simplify the application of the theorem. Assuming that  $n$  is sufficiently large, we have  $|\binom{\alpha}{n}/\binom{\alpha}{n-1}| = |\alpha - n + 1|/n = \frac{n-(\alpha+1)}{n} = 1 - \frac{\alpha+1}{n}$ . We now know that the series converges if  $\alpha + 1 > 1$ , i.e., if  $\alpha > 0$ , and that the series diverges if  $\alpha + 1 < 1$ , i.e., if  $\alpha < 0$ . (The case  $\alpha = 0$  is taken care of, because the infinite series is in fact finite in that case.) We have therefore proved (b).

Abel's Theorem now tells us that if  $\alpha > 0$ ,  $\sum \binom{\alpha}{n} (-1)^n = \lim_{x \rightarrow -1} \sum \binom{\alpha}{n} x^n = \lim_{x \rightarrow -1} (1+x)^\alpha = 0$ , and the series  $\sum \binom{\alpha}{n} x^n$  is uniformly convergent on  $[-1, 0]$ .

Next, an apparent detour: We prove by induction that  $\sum_{n=0}^k \binom{\alpha}{n} (-1)^n = (-1)^k \binom{\alpha-1}{k}$ . The case  $k = 0$  is easily checked:  $\binom{\alpha}{0} = 1$  and  $(-1)^0 \binom{\alpha-1}{0} = 1$ . Assuming that the formula is true for  $k$ , we compute  $\sum_{n=0}^{k+1} \binom{\alpha}{n} (-1)^n = \sum_{n=0}^k \binom{\alpha}{n} (-1)^n + \binom{\alpha}{k+1} (-1)^{k+1} = \binom{\alpha-1}{k} (-1)^k + (-1)^{k+1} \binom{\alpha}{k+1} = (-1)^{k+1} \left[ \binom{\alpha}{k+1} - \binom{\alpha-1}{k} \right] = (-1)^{k+1} \left[ \binom{\alpha-1}{k+1} + \binom{\alpha-1}{k} - \binom{\alpha-1}{k} \right] = (-1)^{k+1} \binom{\alpha-1}{k+1}$ , thereby establishing the inductive hypothesis.

What's the point of all of this? We know that if  $\alpha > 0$ , then  $\lim_{k \rightarrow \infty} \sum_{n=0}^k (-1)^n \binom{\alpha}{n} = 0$ . This equation shows that if  $\alpha > 0$ , then  $\lim_{k \rightarrow \infty} \binom{\alpha-1}{k} = 0$ . Setting  $\beta = \alpha - 1$ , we can say that if  $\beta > -1$ , then  $\lim_{k \rightarrow \infty} \binom{\beta}{k} = 0$ .

If  $\beta > -1$ , then  $\sum \binom{\beta}{n}$  is eventually an alternating series of decreasing terms, so it converges if  $\lim_{n \rightarrow \infty} \binom{\beta}{n} = 0$ , which is exactly what we just demonstrated. We also have  $\sum \binom{-1}{n} = \sum (-1)^n$ , which does not converge. If  $\beta < -1$ , the fact that  $\binom{\beta}{n+1}/\binom{\beta}{n} = \frac{\beta-n}{n+1}$  shows that  $|\binom{\beta}{n+1}| > |\binom{\beta}{n}|$ . That implies that if  $\beta < -1$ ,  $\lim_{n \rightarrow \infty} \binom{\beta}{n} \neq 0$ , showing that  $\sum \binom{\beta}{n}$  does not converge.

Finally, Abel's Theorem says that if  $\beta > -1$ ,  $\sum \binom{\beta}{n} = \lim_{x \rightarrow 1} \sum \binom{\beta}{n} x^n = \lim_{x \rightarrow 1} (1+x)^\beta = 2^\beta$ , and that  $\sum \binom{\beta}{n} x^n$  converges uniformly on  $[0, 1]$ . In the event that  $-1 < \beta < 0$ , standard facts about power series say that the series converges uniformly on  $[-1 + \delta, 0]$  for any  $\delta > 0$ .

5. **10.5.4.** Let  $f(x) = \frac{\sin x}{x}$  on  $[1, \infty)$ . Show that the improper Riemann integral

$$\int_1^{\infty} f(x) dx$$

is convergent. Show that  $f$  is not Lebesgue integrable on  $[1, \infty)$ .

*Answer:* Set  $u = \frac{1}{x}$ ,  $dv = \sin x dx$ ,  $du = -x^{-2} dx$ , and  $v = -\cos x$ , and integrate by parts. We have

$$\int_1^{\infty} \frac{\sin x}{x} dx = \lim_{b \rightarrow \infty} \int_1^b \frac{\sin x}{x} dx = \lim_{b \rightarrow \infty} \left[ \frac{-\cos x}{x} \right]_1^b - \int_1^b \frac{\cos x}{x^2} dx = \cos 1 - \lim_{b \rightarrow \infty} \int_1^b \frac{\cos x}{x^2} dx.$$

We know that  $\int_1^{\infty} x^{-2} dx$  exists. Theorem 10.1.2 tells us that  $\int_1^{\infty} |x^{-2} \cos x| dx$  exists, and then Corollary 10.1.1 implies that  $\int_1^{\infty} x^{-2} \cos x dx$  exists (all as Riemann integrals). Therefore, the given improper Riemann integral is convergent.

However,

$$\int_{(k-1)\pi}^{k\pi} \frac{|\sin x|}{x} dx \geq \frac{1}{k\pi} \int_{(k-1)\pi}^{k\pi} |\sin x| dx = \frac{1}{k\pi} \int_0^{\pi} \sin x dx = \frac{2}{k\pi}.$$

Therefore,

$$\lim_{b \rightarrow \infty} \int_1^b \frac{|\sin x|}{x} dx \geq \lim_{N \rightarrow \infty} \int_{\pi}^{N\pi} \frac{|\sin x|}{x} dx = \lim_{N \rightarrow \infty} \frac{2}{\pi} \left( \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N} \right) = +\infty.$$

Because  $\left| \frac{\sin x}{x} \right|$  is not Lebesgue integrable on  $[1, \infty)$ , we can conclude that  $\frac{\sin x}{x}$  is not Lebesgue integrable on  $[1, \infty)$ .

6. **10.5.10.** (a) Show that

$$\int_0^{\pi} \frac{\sin(n + \frac{1}{2})x}{\sin \frac{x}{2}} dx = \pi.$$

(b) Show that

$$\lim_{\alpha \rightarrow \infty} \int_0^{\pi} \left[ \frac{2}{x} - \frac{1}{\sin \frac{x}{2}} \right] \sin \left( \alpha + \frac{1}{2} \right) x dx = 0.$$

(c) Using the substitution  $t = (\alpha + \frac{1}{2})x$ , show that

$$\int_0^{\infty} \frac{\sin t}{t} dt = \frac{\pi}{2}.$$

*Answer:* (a) Recall that  $D_n(x) = \sum_{k=-n}^n e^{ikx} = \frac{\sin(n + \frac{1}{2})x}{\sin \frac{x}{2}}$ . If we rewrite the sum, we have  $D_n(x) = 1 +$

$\sum_{k=1}^n e^{ikx} + e^{-ikx} = 1 + \sum_{k=1}^n 2 \cos(kx)$ , thereby establishing that  $1 + \sum_{k=1}^n 2 \cos(kx) = \frac{\sin(n + \frac{1}{2})x}{\sin \frac{x}{2}}$  and also that

$$\int_0^{\pi} \frac{\sin(n + \frac{1}{2})x}{\sin \frac{x}{2}} dx = \pi.$$

(b) The function  $\frac{2}{x} - \frac{1}{\sin(x/2)}$  is analytic on  $[0, \pi]$  except perhaps at  $x = 0$ . We simplify:

$$\frac{2}{x} - \frac{1}{\sin(x/2)} = \frac{2 \sin(x/2) - x}{x \sin(x/2)} = \frac{2 \left( \frac{x}{2} - \frac{1}{6} \left( \frac{x}{2} \right)^3 + \cdots \right) - x}{x \left( \frac{x}{2} - \frac{1}{6} \left( \frac{x}{2} \right)^3 + \cdots \right)} = \frac{-\frac{1}{3} \left( \frac{x}{2} \right)^3 + \cdots}{\frac{x^2}{2} - \frac{x^4}{48} + \cdots}$$

After canceling  $x^2/2$  from both numerator and denominator, we see that the denominator begins with  $1+\dots$ , thereby showing that the function is analytic at  $x = 0$ . In particular, the function is Riemann integrable, so we can apply the Riemann-Lebesgue Lemma and conclude that

$$\lim_{\alpha \rightarrow \infty} \int_0^\pi \left[ \frac{2}{x} - \frac{1}{\sin \frac{x}{2}} \right] \sin \left( \alpha + \frac{1}{2} \right) x dx = 0.$$

(c) Restricting  $\alpha$  to be an integer  $n$ , we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_0^\pi \left[ \frac{2}{x} - \frac{1}{\sin \frac{x}{2}} \right] \sin \left( n + \frac{1}{2} \right) x dx &= 0 \\ \lim_{n \rightarrow \infty} \int_0^\pi \left( \frac{2}{x} \right) \sin \left( n + \frac{1}{2} \right) x dx - \lim_{n \rightarrow \infty} \int_0^\pi \left( \frac{1}{\sin \frac{x}{2}} \right) \sin \left( n + \frac{1}{2} \right) x dx &= 0 \\ \lim_{n \rightarrow \infty} \int_0^\pi \left( \frac{2}{x} \right) \sin \left( n + \frac{1}{2} \right) x dx - \pi &= 0 \\ \lim_{n \rightarrow \infty} \int_0^\pi \left( \frac{2}{x} \right) \sin \left( n + \frac{1}{2} \right) x dx &= \pi \end{aligned}$$

Now set  $t = (n + \frac{1}{2})x$ , so  $x = t/(n + \frac{1}{2})$  and  $dx = dt/(n + \frac{1}{2})$ . We have

$$\int_0^\pi \left( \frac{2}{x} \right) \sin \left( n + \frac{1}{2} \right) x dx = \int_0^{(n+\frac{1}{2})\pi} \left( \frac{2}{t} \right) \left( n + \frac{1}{2} \right) \sin t \frac{dt}{n + \frac{1}{2}} = 2 \int_0^{(n+\frac{1}{2})\pi} \frac{\sin t}{t} dt.$$

Returning to the previous computation, we have:

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_0^\pi \left( \frac{2}{x} \right) \sin \left( n + \frac{1}{2} \right) x dx &= \pi \\ \lim_{n \rightarrow \infty} 2 \int_0^{(n+\frac{1}{2})\pi} \frac{\sin t}{t} dt &= \pi \\ \int_0^\infty \frac{\sin t}{t} dt &= \frac{\pi}{2} \end{aligned}$$